Open Court Classics of Science and Philosophy.

THE
GEOMETRICAL LECTURES
OF
ISAAC BARROW

J. M. CHILD

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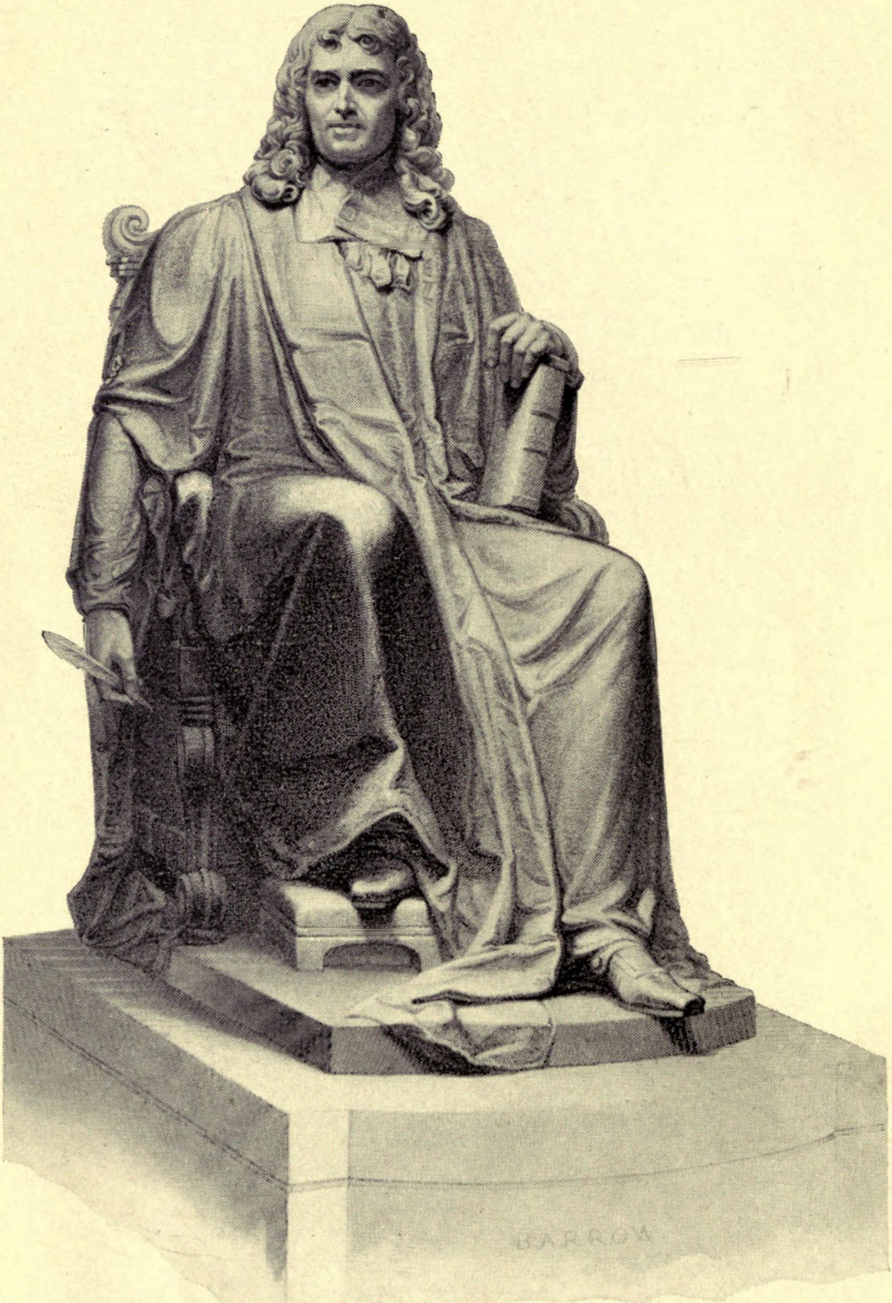
BARROW

The Open Court Series of Classics of Science and Philosophy, No. 3

THE
GEOMETRICAL LECTURES
OF
ISAAC BARROW D.D.

TRANSLATED, WITH NOTES AND PROOFS, AND A DISCUSSION ON THE ADVANCE MADE THEREIN ON THE WORK OF HIS PREDECESSORS IN THE INFINITESIMAL CALCULUS

BY

J. M. CHILD

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CHICAGO AND LONDON
THE OPEN COURT PUBLISHING COMPANY
1916

B3
1916

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LECTIONES Geometricæ;

In quibus (præſertim)
GENERALIA *Curvarum Linearum* SYMPTOMATA
DECLARANTUR.

Auctore ISAACO BARROW Collegii
SS. *Trinitatis* in Acad. *Cantab.* Socio, & *Societatis Regiæ* Sodale.

Οἱ φύσει λογιστικοὶ εἰς πάντα τὰ μαθήματα, ὡς ἔπος εἰπεῖν, ὀξεῖς φαίνονται· οἵ τε βραδεῖς, ἂν ἐν τούτῳ παιδευθῶσι καὶ γυμνάσωνται, κἂν μηδὲν ἄλλο ὠφεληθῶσιν, ὅμως εἴς γε τὸ ὀξύτεροι αὐτοὶ αὑτῶν γίγνεσθαι πάντες ἐπιδιδόασιν. Plato de Repub. VII.

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LONDINI,
Typis *Gulielmi Godbid*, & proſtant venales apud
Johannem Dunmore, & *Octavianum Pulleyn* Juniorem.
M. DC. LXX.

Note the absence of the usual words "Habitæ Cantabrigiæ," which on the title-pages of his other works indicate that the latter were delivered as Lucasian Lectures.—J. M. C.

382569

PREFACE

Isaac Barrow was the first inventor of the Infinitesimal Calculus; Newton got the main idea of it from Barrow by personal communication; and Leibniz also was in some measure indebted to Barrow's work, obtaining confirmation of his own original ideas, and suggestions for their further development, from the copy of Barrow's book that he purchased in 1673.

The above is the ultimate conclusion that I have arrived at, as the result of six months' close study of a single book, my first essay in historical research. By the "Infinitesimal Calculus," I intend "a complete set of standard forms for both the differential and integral sections of the subject, together with rules for their combination, such as for a product, a quotient, or a power of a function; and also a recognition and demonstration of the fact that differentiation and integration are inverse operations."

The case of Newton is to my mind clear enough. Barrow was familiar with the paraboliforms, and tangents and areas connected with them, in from 1655 to 1660 at the very latest; hence he could at this time differentiate and integrate *by his own method* any rational positive power of a variable, and thus also a sum of such powers. He further developed it in the years 1662–3–4, and in the latter year probably had it fairly complete. In this year he communicated to Newton the great secret of his geometrical constructions, as far as it is humanly possible to judge from a collection of tiny scraps of circumstantial evidence; and it was probably this that set Newton to work on an attempt to express everything as a sum of powers of the variable. During *the next year* Newton began to "reflect on his method of fluxions," and actually did produce his *Analysis per Æquationes*. This, though composed in 1666, was not published until 1711.

The case of Leibniz wants more argument that I am in a position at present to give, nor is this the place to give it. I hope to be able to submit this in another place at some future time. The striking points to my mind are that Leibniz bought a copy of Barrow's work in 1673, and was able "to communicate a candid account of his calculus to Newton" in 1677. In this connection, in the face of Leibniz' persistent denial that he received any assistance whatever from Barrow's book, we must bear well in mind Leibniz' twofold idea of the "calculus":—

(i) the freeing of the matter from geometry,
(ii) the adoption of a convenient notation.

Hence, be his denial a mere quibble or a candid statement without any thought of the idea of what the "calculus" really is, it is perfectly certain that on these two points at any rate he derived not the slightest assistance from Barrow's work; for the first of them would be dead against Barrow's practice and instinct, and of *the second Barrow had no knowledge whatever.* These points have made the calculus the powerful instrument that it is, and for this the world has to thank Leibniz; but their inception does not mean the invention of the infinitesimal calculus. This, the epitome of the work of his predecessors, and its completion by his own discoveries until it formed a perfected method of dealing with the problems of tangents and areas for *any curve in general, i.e.* in modern phraseology, the differentiation and integration of any function whatever (such as were known in Barrow's time), must be ascribed to Barrow.

Lest the matter that follows may be considered rambling, and marred by repetitions and other defects, I give first some account of the circumstances that gave rise to this volume. First of all, I was asked by Mr P. E. B. Jourdain to write a short account of Barrow for the *Monist*; the request being accompanied by a first edition copy of Barrow's *Lectiones Opticæ et Geometricæ.* At this time, I do not mind confessing, my only knowledge of Barrow's claim to fame was that he had been "Newton's tutor": a notoriety as unenviable as being known as "Mrs So-and-So's husband." For this article I read, as if for a review, the book that had been sent to me. My attention was arrested by a theorem in which Barrow had *rectified the cycloid*, which I happened to know has usually been ascribed to Sir C. Wren. My interest thus aroused impelled me to make a laborious (for I am no classical scholar) translation of the whole of the geometrical lectures, to see what else I could find. The conclusions I arrived at were sent to the *Monist* for publication; but those who will read the article and this volume will find that in the article I had by no means reached the stage represented by this volume. Later, as I began to still further appreciate what these lectures really meant, I conceived the idea of publishing a full translation of the lectures together with a summary of the work of Barrow's more immediate predecessors, written in the same way from a personal translation of the originals, or at least of all those that I could obtain. On applying to the University Press, Cambridge, through my friend, the Rev. J. B. Lock, I was referred by Professor Hobson to the recent work of Professor Zeuthen. On communicating with Mr Jourdain, I was invited to elaborate my article for the *Monist* into a 200-page volume for the *Open Court Series of Classics*.

I can lay no claim to any great perspicacity in this discovery of mine, if I may call it so; all that follows is due rather to the lack of it, and to the lucky accident that made me (when I could not follow the demonstration) turn one of Barrow's theorems into algebraical geometry. What I found induced me to treat a number of the theorems in the same way. As a result I came to the conclusion that Barrow *had got the calculus*; but I queried even then whether Barrow himself recognized the fact. Only on completing my annotation of the last chapter of this volume, Lect. XII, App. III, did I come to the conclusion that is given as the opening sentence of this Preface; for I then found that a batch of theorems (which I had on first reading noted as very interesting, but not of much service), on careful revision, turned out to be the few missing standard forms, *necessary for completing the set for integration*; and that one of his problems was a practical rule for finding the area under any curve, such as would not yield to the theoretical rules he had given, under the guise of an "inverse-tangent" problem.

The reader will then understand that the conclusion is the effect of a gradual accumulation of evidence (much as a detective picks up clues) on a mind previously blank as regards this matter, and therefore perfectly unbiased. This he will see reflected in the gradual transformation from tentative and imaginative suggestions in the Introduction to direct statements in the notes, which are inset in the text of the latter part of the translation. I have purposely refrained from altering the Introduction, which preserves the form of my article in the *Monist*, to accord with my final ideas, because I feel that with the gradual development thus indicated I shall have a greater chance of carrying my readers with me to my own ultimate conclusion.

The order of writing has been (after the first full translation had been made):—Introduction, Sections I to VIII, excepting III; then the text with notes; then Sections III and IX of the Introduction; and lastly some slight alterations in the whole and Section X.

In Section I, I have given a wholly inadequate account of the work of Barrow's immediate predecessors; but I felt that this could be enlarged at any reader's pleasure, by reference to the standard historical authorities; and that it was hardly any of my business, so long as I slightly expanded my *Monist* article to a sufficiency for the purpose of showing that the time was now ripe for the work of Barrow, Newton, and Leibniz. This, and the next section, have both been taken from the pages of the *Encyclopædia Britannica* (*Times* edition).

The remainder of my argument simply expresses my own, as I have said, gradually formed opinion. I have purposely refrained from consulting any authorities other than the work cited above, the *Bibliotheca Britannica* (for the dates in Section III), and the *Dictionary of National Biography* (for Canon Overton's life of Barrow); but I must acknowledge the service rendered me by the dates and notes in Sotheran's *Price Current of Literature*. The translation too is entirely my own—without any help from the translation by Stone or other assistance—from a first edition of Barrow's work dated 1670.

As regards the text, with my translation beside me, I have to all intents rewritten Barrow's book; although throughout I have adhered fairly closely to Barrow's own words. I have only retained those parts which seemed to me to be absolutely essential for the purpose in hand. Thus the reader will find the first few chapters very much abbreviated, not only in the matter of abridgment, but also in respect of proofs omitted, explanations cut down, and figures left out, whenever this was possible without breaking the continuity. This was necessary in order that room might be found for the critical notes on the theorems, the inclusion of proofs omitted by Barrow, which when given in Barrow's style, and afterwards translated into analysis, had an important bearing on the point as to how he found out the more difficult of his constructions; and lastly for deductions therefrom that point steadily, one after the other, to the fact that Barrow was writing a calculus and *knew that he was inventing a great thing*. I can make no claim to any classical attainments, but I hope the translation will be found correct in almost every particular. In the wording I have adhered to the order in which the original runs, because thereby the old-time flavour is not lost; the most I have done is to alter a passage from the active to the passive or *vice versa*, and occasionally to change the punctuation.

I have used three distinct kinds of type: the most widely spaced type has been used for Barrow's own words; only very occasionally have I inserted anything of my own in this, and then it will be found enclosed in heavy square brackets, that the reader will have no chance of confusing my explanations with the text; the whole of the Introduction, including Barrow's Prefaces, is in the closer type; this type is also used for my critical notes, which are generally given at the end of a lecture, but also sometimes occur at the end of other natural divisions of the work, when it was thought inadvisable to put off the explanation until the end of the lecture. It must be borne in mind that Barrow makes use of parentheses very frequently, so that the reader must understand that only remarks in *heavy square* brackets are mine, those in ordinary round brackets are Barrow's. The small type is used for footnotes only. In the notes I have not hesitated to use the Leibniz notation, because it will probably convey my meaning better; but there was really no absolute necessity for this, Barrow's *a* and *e*, or its modern equivalent, *h* and *k*, would have done quite as well.

I cannot close this Preface without an acknowledgment of my great indebtedness to Mr Jourdain for frequent advice and help; I have had an unlimited call on his wide reading and great historical knowledge; in fact, as Barrow says of Collins, I am hardly doing him justice in calling him my "Mersenne." All the same, I accept full responsibility for any opinions that may seem to be heretical or otherwise out of order. My thanks are also due to Mr Abbott, of Jesus College, Cambridge, for his kind assistance in looking up references that were inaccessible to me.

J. M. CHILD.

DERBY, ENGLAND,
Xmas, 1915.

P.S.—Since this volume has been ready for press, I have consulted several authorities, and, through the kindness of Mr Walter Stott, I have had the opportunity of reading Stone's translation. The result I have set in an appendix at the end of the book. The reader will also find there a solution, by Barrow's methods, of a test question suggested by Mr Jourdain; after examining this I doubt whether any reader will have room for doubt concerning the correctness of my main conclusion. I have also given two specimen pages of Barrow's text and a specimen of his folding plates of diagrams. Also, I have given an example of Barrow's graphical integration of a function; for this I have chosen a function which he could not have integrated theoretically, namely, $1/\sqrt{(1-x^4)}$, between the limits 0 and x; when the upper limit has its maximum value, 1, it is well known that the integral can be expressed in Gamma functions; this was used as a check on the accuracy of the method.

J. M. C.

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INTRODUCTION

I

THE WORK OF BARROW'S GREAT PREDECESSORS

THE beginnings of the Infinitesimal Calculus, in its two main divisions, arose from determinations of areas and volumes, and the finding of tangents to plane curves. The ancients attacked the problems in a strictly geometrical manner, making use of the "method of exhaustions." In modern phraseology, they found "upper and lower limits," as closely equal as possible, between which the quantity to be determined must lie; or, more strictly perhaps, they showed that, if the quantity could be approached from two "sides," on the one side it was always greater than a certain thing, and on the other it was always less; hence it must be finally equal to this thing. This was the method by means of which Archimedes proved most of his discoveries. But there seems to have been some distrust of the method, for we find in many cases that the discoveries are proved by a *reductio ad absurdum*, such as one is familiar with in Euclid. To Apollonius we are indebted for a great many of the properties, and to Archimedes for the measurement, of the conic sections and the solids formed from them by their rotation about an axis.

The first great advance, after the ancients, came in the beginning of the seventeenth century. Galileo (1564–1642) would appear to have led the way, by the introduction of the theory of composition of motions into mechanics;* he also was one of the first to use infinitesimals in geometry, and from the fact that he uses what is equivalent to "virtual velocities" it is to be inferred that the idea of time as the independent variable is due to him. Kepler (1571–1630) was the first to introduce the idea of infinity into geometry and to note that the increment of a variable was evanescent for values of the variable in the immediate neighbourhood of a maximum or minimum; in 1613, an abundant vintage drew his attention to the defective methods in use for estimating the cubical contents of vessels, and his essay on the subject (*Nova Stereometria Doliorum*) entitles him to rank amongst those who made the discovery of the infinitesimal calculus possible. In 1635, Cavalieri published a theory of "indivisibles," in which he considered a line as made up of an infinite number of points, a superficies as composed of a succession of lines, and a solid as a succession of superficies; thus laying the foundation for the "aggregations" of Barrow. Roberval seems to have been the first, or at the least an independent, inventor of the method; but he lost credit for it, because he did not publish it, preferring to keep the method to himself for his own use; this seems to have been quite a usual thing amongst learned men of that time, due perhaps to a certain professional jealousy. The method was severely criticized by contemporaries, especially by Guldin, but Pascal (1623–1662) showed that the method of indivisibles was as rigorous as the method of exhaustions, in fact that they were practically identical. In all probability the progress of mathematical thought is much indebted to this defence by Pascal. Since this method is exactly analogous to the ordinary method of integration, Cavalieri and Roberval have more than a little claim to be regarded as the inventors of at least the one branch of the calculus; *if it were not for the fact* that they only applied it to special cases, and seem to have been unable to generalize it owing to cumbrous algebraical notation, or to have failed to perceive the inner meaning of the method when concealed under a geometrical form. Pascal himself applied the method with great success, but also to special cases only; such as his work on the cycloid. The next step was of a more analytical nature; by the method of indivisibles, Wallis (1616–1703) reduced the determination of many areas and volumes to the calculation of the value of the series $(0^m + 1^m + 2^m + \; . \; . \; . \; n^m)/(n + 1)n^m$, *i.e.* the ratio of the mean of all the terms to the last term, for integral values of n; and later he extended his method, by a theory of interpolation, to fractional values of n. Thus the idea of the Integral

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* See Mach's *Science of Mechanics* for fuller details.

Calculus was in a fairly advanced stage in the days immediately antecedent to Barrow.

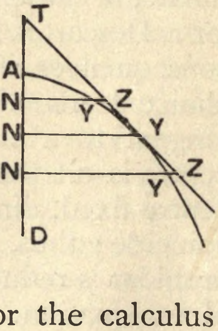
What Cavalieri and Roberval did for the integral calculus, Descartes (1596–1650) accomplished for the differential branch by his work on the application of algebra to geometry. Cartesian coordinates made possible the extension of investigations on the drawing of tangents to special curves to the more general problem for curves of any kind. To this must be added the fact that he *habitually used the index notation*; for this had a very great deal to do with the possibility of Newton's discovery of the general binomial expansion and of many other infinite series. Descartes failed, however, to make any very great progress on his own account in the matter of the drawing of tangents, owing to what I cannot help but call an unfortunate choice of a definition for a tangent to a curve in general. Euclid's circle-tangent definition being more or less hopeless in the general case, Descartes had the choice of three:—

(1) a secant, of which the points of intersection with the curve became coincident;
(2) a prolongation of an element of the curve, which was to be considered as composed of an infinite succession of infinitesimal straight lines;
(3) the direction of the resultant motion, by which the curve might have been described.

Descartes chose the first; I have called this choice unfortunate, because I cannot see that it would have been possible for a Descartes to miss the differential triangle, and all its consequences, if he had chosen the second definition. His choice leads him to the following method of drawing a tangent to a curve in general. Describe a circle, whose centre is on the axis of x, to cut the curve; keeping the centre fixed, diminish the radius until the points of section coincide; thus, by the aid of the equation of the curve, the problem is reduced to finding the condition for equal roots of an equation.

For instance, let $y^2 = 4ax$ be the equation to a parabola, and $(x-p)^2 + y^2 = r^2$ the equation of the circle. Then we have $(x-p)^2 + 4ax = r^2$. If this is a perfect square, $x = p - 2a$; *i.e.* the subtangent is equal to $2a$.

The method, however, is only applicable to a small number of simple cases, owing to algebraical difficulties. In the face of this disability, it is hard to conjecture why Descartes did not make another choice of definition and use the second one given above; for in his rule for the tangents to roulettes, he considers a curve as the ultimate form of a polygon. The third definition, if not originally due to Galileo, was a direct consequence of his conception of the composition of motions; this definition was used by Roberval (1602–1675) and applied successfully to a dozen or so of the well-known curves; in it we have the germ of the method of "fluxions." Thus it is seen that Roberval occupies an almost unique position, in that he took a great part in the work preparatory to the invention of *both branches* of the infinitesimal calculus; a fact that seems to have escaped remark. Fermat (1590–1663) adopted Kepler's notion of the increment of the variable becoming evanescent near a maximum or minimum value, and upon it based his method of drawing tangents. Fermat's method of finding the maximum or minimum value of a function involved the differentiation of any explicit algebraic function, in the *form that appears in any beginner's text-book of to-day* (though Fermat does not seem to have the "function" idea); that is, the maximum or minimum values of $f(x)$ are the roots of $f'(x) = 0$, where $f'(x)$ is the limiting value of $[f(x+h) - f(x)]/h$; only Fermat uses the letter e or E instead of h. Now, if YYY is any curve, wholly concave (or convex) to a straight line AD, TZYZ a tangent to it at the point Y whose ordinate is NY, and the tangent meets AD in T; also, if ordinates NYZ are drawn on either side of NY, cutting the curve in Y and the tangent in Z; then it is plain that the ratio YN : NT is a maximum (or a minimum) when Y is the point of contact of the tangent.



Here then we have all the essentials for the calculus; but only for explicit integral algebraic functions, needing the binomial expansion of Newton, or a general method of rationalization which did not impose too great algebraic difficulties, for their further development; also, on the authority of Poisson, Fermat is placed out of court, in that he also only applied his method to certain special cases. Following the lead of Roberval, Newton subsequently used the third definition of a tangent, and the idea of time as the independent variable, although this was only to insure that one at least of his working variables should increase uniformly. This uniform increase of the independent variable would seem to have been usual for mathematicians of the period and to have persisted for some time; for later we find with Leibniz and the Bernoullis that $d(dy/dx) = (d^2y/dx^2)dx$. Barrow also used time as the independent variable in order that, like Newton, he might insure that one of his variables, a moving point or line or superficies, should proceed uniformly; it is to be noted, however, that this is only in the lectures that were added as an afterthought to the strictly geometrical lectures, and that later this idea becomes altogether subsidiary. Barrow, however, chose his own definition of a tangent, the second of those given above; and to this choice is due in great measure his advance over his predecessors. For his areas and volumes he followed the idea of Cavalieri and Roberval.

Thus we see that in the time of Barrow, Newton, and Leibniz the ground had been surveyed, and in many directions levelled; all the material was at hand, and it only wanted the master mind to "finish the job." This was possible in two directions, by geometry or by analysis; each method wanted a master mind of a totally different type, and the men were forthcoming. For geometry, Barrow; for analysis, Newton, and Leibniz with his inspiration in the matter of the application of the simple and convenient notation of his calculus of finite differences to infinitesimals and to geometry. With all due honour to these three mathematical giants, however, I venture to assert that their discoveries would have been well-nigh impossible to them if they had lived a hundred years earlier; with the possible exception of Barrow, who, being a geometer, was more dependent on the ancients and less on the moderns of his time than were the two analysts, they would have been sadly hampered but for the preliminary work of Descartes and the others I have mentioned (and some I have not—such as Oughtred), but especially Descartes.

II

LIFE OF BARROW, AND HIS CONNECTION WITH NEWTON

Isaac Barrow was born in 1630, the son of a linen-draper in London. He was first sent to the Charterhouse School, where inattention and a predilection for fighting created a bad impression; his father was overheard to say (*pray*, according to one account) that "if it pleased God to take one of his children, he could best spare Isaac." Later, he seems to have turned over a new leaf, and in 1643 we find him entered at St Peter's College, Cambridge, and afterwards at Trinity. Having now become exceedingly studious, he made considerable progress in literature, natural philosophy, anatomy, botany, and chemistry—the three last with a view to medicine as a profession,—and later in chronology, geometry, and astronomy. He then proceeded on a sort of "Grand Tour" through France, Italy, to Smyrna, Constantinople, back to Venice, and then home through Germany and Holland. His stay in Constantinople had a great influence on his after life; for he here studied the works of Chrysostom, and thus had his thoughts turned to divinity. But for this, his great advance on the work of his predecessors in the matter of the infinitesimal calculus might have been developed to such an extent that the name of Barrow would have been inscribed on the roll of fame as at least the equal of his mighty pupil Newton.

Immediately on his return to England he was ordained, and a year later, at the age of thirty, he was appointed to the Greek professorship at Cambridge; his inaugural lectures were on the subject of the *Rhetoric* of Aristotle, and this choice had also a distinct effect on his later mathematical work. In 1662, two years later, he was appointed Professor of Geometry in Gresham College; and in the following year he was elected to the Lucasian Chair of Mathematics, just founded at Cambridge. This professorship he held for five years, and his office created the occasion for his *Lectiones Mathematicæ*, which were delivered in the years 1664–5–6 (*Habitæ Cantabrigiæ*). These lectures were published, according to Prof. Benjamin Williamson (*Encyc. Brit.*

(*Times* edition), Art. on Infinitesimal Calculus) in 1670; this, however, is wrong: they were not published until 1683, under the title of *Lectiones Mathematicæ*. What was published in 1670 was the *Lectiones Opticæ et Geometricæ*; the *Lectiones Mathematicæ* were philosophical lectures on the fundamentals of mathematics and did not have much bearing on the infinitesimal calculus. They were followed by the *Lectiones Opticæ* and lectures on the works of Archimedes, Apollonius, and Theodosius; in what order these were delivered in the schools of the University I have been unable to find out; but the former were published in 1669, "Imprimatur" having been granted in March 1668, so that it was probable that they were the professorial lectures for 1667; thus the latter would have been delivered in 1668, though they were not published until 1675, and then probably by Collins. The great work, *Lectiones Geometricæ*, did not appear as a separate publication at first: as stated above, it was issued bound up with the second edition of the *Lectiones Opticæ*; and, judging from the fact that there does not, according to the above dates, appear to have been any time for their public delivery as Lucasian Lectures, since Imprimatur was granted for the combined edition in 1669; also from the fact that Barrow's Preface speaks of six out of the thirteen lectures as "matters left over from the Optics," which he was induced to complete to form a separate work; also from the most conclusive fact of all, that on the title-page of the *Lectiones Geometricæ* there is no mention at all of the usual notice "Habitæ Cantabrigiæ";—judging from these facts, I do not believe *that the* "*Lectiones Geometricæ*" *were delivered as Lucasian Lectures*. Should this be so, it would clear up a good many difficulties; it would corroborate my suggestion that they were for a great part evolved during his professorship at Gresham College; also it would make it almost certain that they would have been given as internal college lectures, and that Newton would have heard them in 1663–1664.

Now, it was in 1664 that Barrow first came into close personal contact with Newton; for in that year, he examined Newton in Euclid, as one of the subjects for a mathematical scholarship at Trinity College, of which Newton had been a subsizar for three years; and it was due to Barrow's report that Newton was led to study the *Elements* more carefully and to form a better estimate of their value. The connection once started must have developed at a great pace, for not only does Barrow secure the succession of Newton to the Lucasian chair, when he relinquished it in 1669, but he commits the publication of his *Lectiones Opticæ* to the foster care of Newton and Collins. He himself had now determined to devote the rest of his life to divinity entirely; in 1670 he was created a Doctor of Divinity, in 1672 he succeeded Dr Pearson as Master of Trinity, in 1675 he was chosen Vice-Chancellor of the University; and in 1677 he died, and was buried in Westminster Abbey, where a monument, surmounted by his bust, was soon afterwards erected to his memory by his friends and admirers.

III

THE WORKS OF BARROW

Barrow was a very voluminous writer. On inquiring of the Librarian of the Cambridge University Library whether he could supply me with a complete list of the works of Barrow in order of publication, I was informed that the complete list occupied *four columns in the British Museum Catalogue*! This of course would include his theological works, the several different editions, and the translations of his Latin works. The following list of his mathematical works, such as are important for the matter in hand only, is taken from the *Bibliotheca Britannica* (by Robert Watt, Edinburgh, 1824):—

1.	*Euclid's Elements*,	Camb. 1655.
2.	*Euclid's Data*,	Camb. 1657.
3.	*Lectiones Opticorum Phenomenōn*,	Lond. 1669.
4.	*Lectiones Opticæ et Geometricæ*,	Lond. 1670
	(in 2 vols., 1674; trans., Edmond Stone, 1735).	
5.	*Lectiones Mathematicæ*,	Lond. 1683.

This list makes it absolutely certain that Williamson is wrong in stating that the lectures in geometry were published under the title of "Mathematical Lectures." This, however, is not of much consequence; the important point in the list, assuming it to be perfectly correct as it stands, is that the lectures on Optics were first published separately in 1669. In the following year they were reissued in a revised form with the addition of the lectures on geometry.

The above books were all in Latin and have been translated by different people at one time or another.

IV

ESTIMATE OF BARROW'S GENIUS

The writer of the article on "Barrow, Isaac," in the ninth (*Times*) edition of the *Encyclopædia Britannica*, from which most of the details in Section II have been taken, remarks :—

"By his English contemporaries Barrow was considered a mathematician second only to Newton. Continental writers do not place him so high, and *their judgment is probably the more correct one.*"

Founding my opinion on the *Lectiones Geometricæ* alone, I fail to see the reasonableness of the remark I have italicized. Of course, it was only natural that contemporary continental mathematicians should belittle Barrow, since they claimed for Fermat and Leibniz the invention of the infinitesimal calculus before Newton, and did not wish to have to consider in Barrow an even prior claimant. We see that his own countrymen placed him on a very high level; and surely the only way to obtain a really adequate opinion of a scientist's worth is to accept the unbiased opinion that has been expressed by his contemporaries, who were aware of all the facts and conditions of the case; or, failing that, to try to form an unbiased opinion for ourselves, in the position of his contemporaries. An obvious deduction may be drawn from the controversy between Newton and Hooke; the opinion of Barrow's own countrymen would not be likely to err on the side of over-appreciation, unless his genius was great enough to outweigh the more or less natural jealousy that ever did and ever will exist amongst great men occupied on the same investigations. Most modern criticism of ancient writers is apt to fail, because it is in the hands of the experts; perhaps to some degree this must be so, yet you would hardly allow a K.C. to be a fitting man for a jury.

Criticism by experts, unless they are themselves giants like unto the men whose works they criticize, compares, perhaps unconsciously, their discoveries with facts that are now common knowledge, instead of considering only and solely *the advance made upon what was then common knowledge.* Thus the skilled designers of the wonderful electric engines of to-day are but as pigmies compared with such giants as a Faraday.

Further, in the case of Barrow, there are several other things to be taken into account. We must consider his disposition, his training, his changes of intention with regard to a career, the accident of his connection with such a man as Newton, the circumstances brought about by the work of his immediate predecessors, and the ripeness of the time for his discoveries.

His disposition was pugnacious, though not without a touch of humour; there are many indications in the *Lectiones Geometricæ* alone of an inclination to what I may call, for lack of a better term, a certain contributory laziness; in this way he was somewhat like Fermat, with his usual "I have got a very beautiful proof of . . . : if you wish, I will send it to you; but I dare say you will be able to find it for yourself"; many of Barrow's most ingenious theorems, one or two of his most far-reaching ones, are left without proof, though he states that they are easily deduced from what has gone before. He evidently knows the importance of his discoveries; in one place he remarks that a certain set of theorems are a "mine of information, in which should anyone investigate and explore, he will find very many things of this kind. Let him do so who must, or if it pleases him." He omits the proof of a certain theorem, which he states has been very useful to him *repeatedly*; and no wonder it has, for it turns out to be the equivalent to the differentiation of a *quotient*; and yet he says, "It is sufficient for me to mention this, and indeed I intend to stop here for a while." It is not at all strange that the work of such a man should come to be underrated.

His pugnacity is shown in the main object pervading the whole of the *Lectiones Geometricæ*; he sets out with the one express intention of simplifying and generalizing the existing methods of drawing tangents to curves of all kinds and of finding areas and volumes; there is distinct humour in his glee at "wiping the eye" of some other geometer, ancient or modern, whose solution of some particular problem he has not only generalized but simplified.

"Gregory St Vincent gave this, but (if I remember rightly) proved with wearisome prolixity."

"Hence it follows immediately that all curves of this kind are touched at any one point by one straight line only. . . . Euclid proved this as a special case for the circle, Apollonius for the conic sections, and other persons in the case of other curves."

His early training was promiscuous, and could have had no other effect than to have fostered an inclination to leave others to finish what he had begun. His Greek professorship and his study of Aristotle would tend to make him a confirmed geometrician, revelling in the "elegant solution" and more or less despising Cartesian analysis because of its then (frequently) cumbersome work, and using it only with certain qualms of doubt as to its absolute rigour. For instance, he almost apologizes for inserting, at the very end of Lecture X, which ends the part of the work devoted to the equivalent of the differential calculus, his "a and e" method—the prototype of the "h and k" method of the ordinary text-books of to-day.

Another light is thrown on the matter of Cartesian geometry, or rather its applications, by Lecture VI; in this, for the purpose of establishing lemmas to be used later, Barrow gives fairly lengthy proofs that

$$(1)\ my \pm xy = mx^2/b, \qquad (2)\ \pm yx + gx - my = mx^2/r$$

represent hyperbolas, instead of merely stating the fact on account of the factorizing of $mx^2/b \mp xy$, $mx^2/r \pm xy$. The lengthiness of these proofs is to a great extent due to the fact that, although the appearance of the work is algebraical, the reasoning is almost purely geometrical. It is also to be noted that the index notation is rarely used, at least not till very late in the book in places where he could do nothing else, although Wallis had used even fractional indices a dozen years before. In a later lecture we have the truly terrifying equation $(rrkk - rrff + 2fmpa)/kk = (rrmm + 2fmpa)/kk$.

Again we must note the fact that all Barrow's work, without exception, was geometry, although it is fairly evident that he used algebra for his own purposes.

From the above, it is quite easy to see a reason why Barrow should not have turned his work to greater account; but in estimating his genius one must make allowance for this disability in, or dislike for, algebraic geometry, read into his work what could have been got out of it (what I am certain both Newton and Leibniz got out of it), and not stop short at just what was actually published. It must chiefly be remembered that these old geometers could use their geometrical facts far more readily and surely than many mathematicians of the present day can use their analysis. As a justification of the extremely high estimate I have formed, from the *Lectiones Geometricæ* alone, of Barrow's genius, I call the attention of the reader to the list of analytical equivalents of Barrow's theorems given on page 30, if he has not the patience to wade through the running commentary which stands instead of a full translation of this book.

V

THE SOURCES OF BARROW'S IDEAS

There is too strong a resemblance between the methods to leave room for doubt that Barrow owed much of his idea of *integration* to Galileo and Cavalieri (or Roberval, if you will). On the question as to the sources from which he derived his notions on differentiation there is considerably more difficulty in deciding; and the comparatively narrow range of my reading makes me diffident in writing anything that may be considered dogmatic on this point; and yet if I do not do so, I shall be in danger of not getting a fair hearing. The following remarks must therefore be considered in the nature of the plea of a "counsel for the defence," who believes absolutely in his client's case; or as suggestions that possibly, even if not probably, come very near to the truth.

The general opinion would seem to be that Barrow was a mere improver on Fermat. But Barrow was conscientious to a fault; and if we are to believe in his honesty, the source of his ideas could not have been Fermat. For Barrow religiously gives references to the ancient and contemporary mathematicians whose work he quotes. These references include Cartesius, Hugenius, Galilæus, Gregorius a St Vincentio, Gregorius Aberdonensis, Wallis, and many others, with Euclides, Aristoteles, Archimedes, Apollonius, among the ancients; but, as far as I can find, no mention is made of Fermat in any place; nor does Barrow use Fermat's idea of determining the tangent algebraically by consideration of a maximum or minimum; these points entirely contradict the notion that he was a mere improver on Fermat, which seems to have arisen because Barrow uses the same letter, e, for his increment of x, and only adds another, a, to signify the increment of y. I suggest that this was only a coincidence; that both adopted the letter e (Fermat seems to have used the capital E) as being the initial letter of the word *excess*, whilst Barrow in addition used the letter a, the initial letter of the word *additional*; if he was a mere improver on Fermat, the improvement was a huge one, for it enabled Barrow to handle, without the algebraical difficulties of Fermat, *implicit* functions as well as explicit functions. On the other hand, Barrow may have got the notion of using arithmetic and geometric means, with which he performs some wonders, from Fermat, who apparently was the first to use them, though by Barrow's time they were fairly common property, being the basis of all systems of logarithms; and Barrow's condition of tangency was so similar to the method of Fermat that, while he could not very well use any other condition with his choice of the definition of a tangent, Barrow may have deliberately omitted any reference to Fermat, for fear that thereby he might, by the reference alone, provoke accusations of plagiarism. As I have already remarked, there is a distinct admiration for the work of Galileo, and the idea of time as the independent variable obsesses the first few lectures; however, he simply intended this as a criterion by means of which he could be sure that one of his variables increased uniformly, or in certain of his theorems in the later parts that he might consider his y as a function of a function; but in most of the later lectures the idea of time becomes quite insignificant. This is, of course, explained by the fact that the original draft of the geometrical lectures consisted only of the lectures numbered VI to XII (including the appendices, with the possible exception of Appendix 3 to Lect. XII); for we read in the Preface that Barrow "falling in with his wishes (I will not say very willingly) added the first five lectures." The word "his" refers to *Librarius*, which—for lack of a better word, or of *editor*, which I do not like—I have translated as *the publisher*; but I think it refers to Collins, for, in Barrow's words, "John Collins looked after the publication."

The opinion I have formed is that the idea of the differential triangle, upon which all attention seems (quite wrongly) to be focussed, when considering the work of Barrow, was altogether his own *original concept*; and to call it a mere improvement on Fermat's method, in that he uses two increments instead of one, is absurd. The discovery was the outcome of Barrow's definition of a tangent, wholly and solely; and the method of Fermat did not consider this.

The mental picture that I form of Barrow, and of the events that led to this discovery, amongst others far more important, is that of the Professor of Geometry at Gresham College, who has to deliver lectures on his subject; he reads up all that he can lay his hands on, decides that it is all very decent stuff of a sort, yet pugnaciously determines that he can and will "go one better." In the course of his researches, he is led from one thing to another until he comes to the paraboliform construction of Lect. IX, § 4, perceives its usefulness and inner meaning, and immediately conceives the idea of the differential triangle. I think if any reader compares the two figures above, Fig. A used for his construction of the paraboliforms, Fig. B for the differential triangle, he will no longer inquire

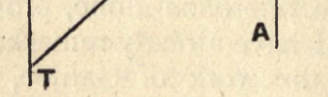


Fig. A.

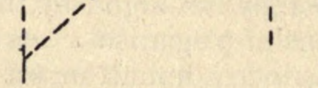


Fig. B.

for the source of Barrow's idea, unless perhaps he may prefer to refer it to Lect. X, § 11.

Personally, I have no doubt that it was a flash of inspiration, suggested by the first figure; and that it was Barrow's luck to have first of all had occasion to draw that figure, and secondly to have had the genius to note its significance and be able to follow up the clue thus afforded. As further corroborative evidence that Barrow's ideas were in the main his own creations, we have the facts that he was alone in using *habitually* the idea of a curve being a succession of an infinite number of infinitely short straight lines, the prolongation of any one representing the tangent at the point on the curve for which the straight line, or either end of of it, stood; also that he could not see any difference between indefinitely narrow rectangles and straight lines as the constituents of an area. If his methods had required it, which they did not, he would no doubt have proved rigorously that the error could be made as small as he pleased by making the number of parts, into which he had divided his area, large enough; this was indeed the substance of Pascal's defence of Cavalieri's method of "indivisibles," and the idea is used in Lect. XII, App. II, § 6.

I have remarked that, in considering the work of Barrow, all attention seems to be quite wrongly focussed on the differential triangle. I hope to convince readers of this volume that the differential triangle was only an important side-issue in the *Lectiones Geometricæ*; certainly Barrow only considered it as such. Barrow really had, concealed under the geometrical form that was *his* method, *a complete treatise on the elements of the calculus.*

The question may then be asked why, if all this is true, did Barrow not finish the work he had begun; and the answer, I take it, is inseparably bound up with the peculiar disposition of Barrow, his growing desire to forsake mathematics for divinity, and the accident of having first as his pupil and afterwards as his co-worker, and one in close personal contact with him, a man like Newton, whose analytical mind was so peculiarly adapted to the task of carrying to a successful conclusion those matters which Barrow saw could not be developed to anything like the extent by his own geometrical method. One writer has stated that the great genius of Barrow must be admitted, if only for the fact that he recognized in the early days of Newton's career the genius of the man, his pupil, that was afterwards to overshadow him. Also, if I fail to make out my contention that Barrow's ideas were in the main original, the same remark can with justice be applied to him that William Wallace in similar circumstances applied to Descartes, that if it were true that he borrowed his ideas on algebra from others, this fact, "would only illustrate the genius of the man who could pick out from other works all that was productive, and state it with a lucidity that makes it look his own discovery"; for the lucidity is there all right in this work of Barrow, only it wants translating into analytical language before it can be readily grasped by anyone but a geometer.

VI

MUTUAL INFLUENCE OF NEWTON AND BARROW

I can image that Barrow's interest, as a confirmed geometer, would have been first aroused by young Newton's poor show in his scholarship paper on Euclid. This was in April 1664, the year of the delivery of Barrow's first lectures as Lucasian Professor, and, according to Newton's own words, just about the time that he, Newton, discovered his method of infinite series, led thereto by his reading of the works of Descartes and Wallis. Newton no doubt attended these lectures of Barrow, and the probability is that he would have shown Barrow his work on infinite series; for this would seem to have been the etiquette or custom of the time; for we know that in 1669 Newton communicated to Collins through Barrow a compendium of his work on fluxions (note that this is the year of the preparation for press of the *Lectiones Opticæ et Geometricæ*). Barrow could not help being struck by the incongruity (to him) of a man of Newton's calibre not appreciating Euclid to the full; at the same time the one great mind would be drawn to the other, and the connection thus started would have ripened inevitably. I suggest as a consequence that Barrow would show Newton his own geometry, Newton would naturally ask Barrow to explain how he had got the idea for some of his more difficult constructions, and Barrow would let him into the secret. "I find out the constructions by this little list of rules, and methods for combining them." "But, my dear sir, the rules are far more valuable than the mere finding of the tangents or the areas." "All right, my boy, if you think so, you are welcome to them, to make what you like of, or what you can; only do not say you got them from me, I'll stick to my geometry."* This was probably the occasion when Newton persuaded Barrow that the differential triangle was more general than all his other theorems put together; also later when the Geometry was being got ready for press, Newton probably asked Barrow to produce from his stock of theorems others necessary to complete his, Barrow's, Calculus, the result being the appendices to Lect. XII.

The rest of the argument is a matter of dates. Barrow was Professor of Greek from 1660 to 1662, then Professor of Geometry at Gresham College from 1662 to 1664, and Lucasian Professor from 1664 to 1669; Newton was a member of Trinity College from 1661, and was in residence until he was forced from Cambridge by the plague in the summer of 1665; from manuscript notes in Newton's handwriting, it was probably during, and owing to, this enforced absence from Cambridge (and, I suggest, away from the geometrical influence of Barrow) that he began to develop the method of fluxions (probably in accordance with some such permission from Barrow as that suggested in the purely imaginative interview above).

The similarity of the two methods of Barrow and Newton is far too close to admit of them being anything else but the outcome of one single idea; and I argue from the dates given above that Barrow had developed most of his geometry from the researches begun for the necessities of lectures at Gresham College. We know that Barrow's work on the difficult theorems and problems of Archimedes was largely a suggestion of a kind of analysis by which they were reduced to their simple component problems. What is then more likely than that this is an intentional or unintentional cryptogrammatic key to Barrow's own method? I suggest that it is more than likely,—IT IS. As I said, the similarity of the two methods of Newton and Barrow is very striking.

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* Of course this is imaginative retrospective prophecy; I beg that no one will take the inverted commas to signify quotations.

For the fluxional method the procedure is as follows:—

(1) Substitute $x+\dot{x}o$ for x and $y+\dot{y}o$ for y in the given equation containing the fluents x and y.

(2) Subtract the original equation, and divide through by o.

(3) Regard o as an evanescent quantity, and neglect o and its powers.

Barrow's rules, in altered order to correspond, are:—

(2) After the equation has been formed (Newton's rule 1), reject all terms consisting of letters denoting constant or determined quantities, or terms which do not contain a or e (which are equivalent to Newton's $\dot{y}o$ and $\dot{x}o$ respectively); for these terms brought over to one side of the equation will always be equal to zero (Newton's rule 2, first part).

(1) In the calculation, omit all terms containing a power of a or e, or products of these letters; for these are of no value (Newton's rule 2, second part, and rule 3).

(3) Now substitute m (the ordinate) for a, and t (the subtangent) for e. (This corresponds with Newton's next step, the obtaining of the ratio $\dot{x} : \dot{y}$, which is exactly the same as Barrow's $e : a$.)

The only difference is that Barrow's way is the more suited to his geometrical purpose of finding the "quantity of the subtangent," and Newton's method is peculiarly adapted to analytical work, especially in problems on motion. Barrow left his method as it stood, though probably using it freely (mark the word *usitatum* on page 119, which is a *frequentative* derivative of *utor*, I use) to obtain hints for his tangent problems, but not thinking much of it as a method compared with a strictly geometrical method; yet admitting it into his work, on the advice of a friend, on account of its *generality*. On the other hand, Newton perceived at once the immense possibilities of the analytical methods introduced by Descartes, and developed the idea on his own lines, to suit his own purposes.

There is still another possibility. In the Preface to the Optics, we read that "as delicate mothers are wont, I committed to the foster care of friends, not unwillingly, *my discarded child.*" . . . These two friends Barrow mentions by name: "*Isaac Newton* . . . (a man of exceptional ability and remarkable skill) has revised the copy, warning me of many things to be corrected, and adding some things from his own work." . . . Newton's additions were probably confined to a great extent to the Optics only ; but the geometrical lectures (seven of them at least) were originally designed as supplementary to the Optics, and would be also looked over by Newton when the combined publication was being prepared. . . . "John Collins has attended to the publication." Hence, it is just possible that Newton showed Barrow his method of fluxions first, and Barrow inserted it in his own way ; this supposition would provide an easy explanation of the treatment accorded to the batch of theorems that form the third appendix to Lect. XII ; they seem to be hastily scrambled together, compared with the orderly treatment of the rest of the book, and are without demonstration ; and this, although they form a necessary complement for the completion of the standard forms and rules of procedure. I say that this is possible, but I do not think it is at all probable ; for it is to be noted that Barrow's description of the method is in the *first person singular* (although, when giving the reason for its introduction, he says "frequently used by *us*") ; and remembering the authentic accounts of Barrow's conscientious honesty, and also judging by the later work of Newton, I think that the only alternative to be considered is that first given. Also, if that is accepted, we have a natural explanation of the lack of what I call the true appreciation of Barrow's genius. Barrow could see the limitations imposed by his own geometrical methods (none so well as he, naturally, being probably helped to this conclusion by his discussions with Newton) ; he felt that the correct development of his idea was on purely analytical lines, he recognised his own disability in that direction and the peculiar aptness of Newton's genius for the task, and, lastly, the growing desire to forsake mathematics for divinity made him only too willing to hand over to the foster care of Newton and Collins his discarded child "to be led out and set forth as *might seem good to them*." "Carte blanche" of such a sweeping character very often has exactly the opposite effect to that which is intended ; and so probably Newton and Collins forbore to make any serious alterations or additions, out of respect for Barrow ; for although the allusion to the revision properly applies only to the Optics, it may fairly be assumed that it would be extended to the Geometry as well; and if not then, at any rate later, for, quoting a quotation by Canon Overton in the *Dictionary of National Biography* (source of the quotation not stated), which refers to Barrow's pique at the poor reception that was accorded to the geometrical lectures—and does not this show the high opinion that Barrow had of them himself, and lend colour to my suggestion that they were never delivered as Lucasian Lectures?; also note his remark in the Preface, given later, "The other seven, as I said, *I expose more freely to your view, hoping that there is nothing in them that it will displease the erudite to see*,"—"When they had been some time in the world, having heard of a very few who had read and *considered them thoroughly*, the little relish that such things met with helped to loose him more from those speculations and heighten his attention to the studies of morality and divinity." Does not this read like the disgust at people forsaking the legitimate methods of geometry for "such unsatisfactory stuff (as I have suggested that Barrow would consider it) as analysis"?

Who can say the form these lectures might have taken if there had been no Newton; or if Barrow had taken kindly to Cartesian geometry; or what a second edition, "revised and enlarged," might have contained, if Barrow on his return to Cambridge as Master of Trinity and Vice-Chancellor had had the energy or the inclination to have made one; or if Newton had made a treatise of it, instead of a reprint of "Scholastic Lectures," as Barrow warns his readers that it is, and such as the edition of 1674 in two volumes probably was? But Barrow died only a few years later, Newton was far too occupied with other matters, and Collins seems to have passed out of the picture, even if he had been the equal of the other two.

VII

DESCRIPTION OF THE BOOK FROM WHICH THE TRANSLATION HAS BEEN MADE

The running commentary which follows is a précis of a full translation of a book in the Cambridge Library. In one volume, bound in strong yellow calf, are the two works, the *Lectiones Opticæ et Geometricæ*; the title-page of the first bears the date MDCLXIX, that of the second the date MDCLXX, whilst "Imprimatur" was granted on 22nd March 1669; this points to its being one of the original combined editions, No. 4 of the list in Section III of this Introduction. On the title-page of the Optics there is a line which reads, "To which are annexed a few geometrical lectures," agreeing with the remark in the preface to the geometrical section that originally there were only seven geometrical lectures that were intended to be published as supplements of the Optics, instead of the thirteen of which the section is composed. For in all probability this title-page is that of the first edition of the Optics, but the *Librarius*, whoever he may have been, persuaded Barrow to leave the seven lectures out, enlarge them to form a separate work, and to publish them as such in combination with the Optics, as we see, in 1670; and by an oversight the title-page remained uncorrected.

Of prefaces there are three, one being more properly an introduction, explaining the plan and scope of the originally designed "XVIII Lectures on Optics" and the supplementary seven geometrical lectures; this is in the same type as, and immediately in front of, the Optics. The other two are true prefaces or "Letters to the reader"; they are in italics: a full translation of both is given later.

On a fly-leaf in front of the Optics is a list of symbols of abbreviation as used by Barrow; as these cover the two sections and are not repeated in front of the geometrical section, they furnish additional evidence that the book I have used is one of the first combined editions. The similarity of the symbols used by Barrow to those used at the present day, *to stand for quite different things*, does not simplify the task of a modern reader. This is especially the case with the signs for "greater than" and "less than," where the "openings" of the signs face the reverse way to that which is now usual; another point which might lead to error by a casual reader who had not happened to notice the list of abbreviations, is the use of the plus sign between two ratios to stand for the ratio compounded from them, *i.e.* for multiplication; the minus sign does not, however, stand for the ratio of two ratios, *i.e.* for division, the ease with which the argument may be followed is also not by any means increased by Barrow's plan of running his work on in one continuous stream (paper was dear in those days), with intermediate steps in brackets; and this is made still worse by the use of the "full stop" as a sign of a ratio (division) instead of as a sign of a rectangle (multiplication); thus $DH . HO :: (DL . LN :: DL - DH . LN - HO :: LH . LB ::) LH . HK$ stands, in modern symbols, for the extended statement

$$DH : HO = DL : LN = (DL - DH) : (LN - HO) = LH : HB$$
$$\therefore \quad DH : HO = LH : LB = LH : HK;$$

whereas $DL \times LK - LH \times HK = KO \times LH - HK \times LH$, on the contrary, means, as is usual at present, $DL . HK - LH . HK = KO . LH - HK . LH$, the minus sign thus being a weaker bond than that of multiplication, but a stronger bond than that of ratio or division. Barrow's list of symbols, in full, is:—

"For the sake of brevity certain signs are used, the meaning of which is here subjoined.

$A + B$	that is,	A and B taken together.
$A - B$		A, B being taken away.
$A -: B$		The difference of A and B.
$A \times B$		A multiplied by, or led into, B.
$\frac{A}{B}$		A divided by B, or applied to B.
$A = B$		A is equal to B.
$A \sqsubset B$		A is greater than B.
$A \sqsupset B$		A is less than B.
$A . B :: C . D$		A bears to B the same ratio as C to D.
A, B, C, D ∺		A, B, C, D are in continued proportion.
$A . B \sqsubset C . D$		A to B is greater than C to D.
$A . B \sqsupset C . D$		A to B is less than C to D.
$A . B + C . D \left.\begin{matrix} = \\ \sqsubset \\ \sqsupset \end{matrix}\right\} M . N$		$\left.\begin{matrix}\text{The ratios} \\ \text{A to B, C to D} \\ \text{compounded}\end{matrix}\right\} \text{are} \left\{\begin{matrix}\text{equal to} \\ \text{greater than} \\ \text{less than}\end{matrix}\right\} \text{M to N.}$
Aq		The square on A.
$\surd A$		The side, or square root of, A.
Ac		The cube of A.
$\surd Aq + Bq$		The side of the square made up of the square of A and the square of B.

Other abbreviations, if there are any, the reader will recognise, by easy conjecture, especially as I have used very little analysis."

The style of the text, as one would expect from a Barrow, is "classical"; that is, full of long involved sentences, phrases such as "through all of a straight line points," general inversion of order to enable the sense to run on, use of the relative instead of the demonstrative, and so on; all agreeing with what is but an indistinct memory (thank goodness!) of my trials and troubles as a boy over Cicero, *De Senectute*, *De Amicitia*, and such-like, studied (?), by the way, in Newton's old school at Grantham in Lincolnshire.

In this way there is a striking difference between the style of Barrow and the straightforward Latin of Newton's *Principia*, as it stands in my Latin edition of 1822, by Le Seur and Jacquier. My classical attainments are, however, so slight that, in looking for possible additions by Newton, I have preferred to rely on my proof-reading experience in the matter of punctuation. The strong point in Barrow's somewhat awful punctuation is the use of the semicolon, combined with the long involved sentence, and the frequent interpolation of arguments, sometimes running to a dozen lines, in parentheses; Newton makes use of the short concise sentence, and rarely uses the semicolon, nor indeed does he use the colon to any great extent. Of course I do not know how much the printer had to do with the punctuation in those days, but imagine this distinction was a very great matter of the author. Comparing two analogous passages, from each author, of about 200–250 words, we get the following table:—

Barrow.	Newton.	
11	15	commas
10	5	semicolons
None	3	colons
5	10	full-stops
4	None	parentheses

This contrast is striking enough for all practical purposes; in addition, Barrow starts three of his five sentences with a relative, whilst Newton does not do this once in his ten.

Using this idea, I failed to find anything that could, with any probability, be ascribed to Newton.

Lastly, one strong feature in the book is the continued use of the paraboliforms as auxiliary curves; this corroborates my contention that Barrow fully appreciated the importance and inner meaning of his theorem, or rather construction (see note to Lect. IX, § 4); that is, he uses it in precisely the same way as the analytical mathematician uses its equivalent, the approximation to the binomial and the differentiation of a fractional power of a variable, as a foundation of all his work.

Although there are two fairly long lists of errata, most probably due to Newton, there are still a great number of misprints; the diagrams are, however, uniformly good, there being no omissions of important letters and only one or two slips in the whole set of 200, one of these evidently being the fault of the engraver; nevertheless they might have been much clearer if Barrow had not been in the habit of using one diagram for a whole batch of allied theorems, thereby having to make the diagram rather complicated in order to get all the curves and lines necessary for the whole batch of theorems on the one figure, whilst only using some of them for each separate theorem. In the text which follows this introduction, only those figures have been retained that were absolutely essential.

There is a book-plate bearing a medallion of George I and the words "MUNIFICENTIA REGIA 1715" which points out that the book I have was one of the 30,000 volumes of books and manuscripts comprised in the library of Bishop Moore of Norwich, which was presented to the Cambridge University Library in 1715 by George I, as an acknowledgment of a loyal address sent up by the University to the king on his accession. It may have come into his possession as a personal gift from Barrow; at any rate, there is an inscription on the first fly-leaf, "A gift from the author." I am unable to ascertain whether Moore was a student at Cambridge at the date of the publication of these lectures, but the date of his birth (1646) would have made him twenty-four years of age at the time, and this supposition would explain the presence of a four-line Latin verse (Barrow had a weakness for turning things into Latin or Greek verse) on the back of the title-page of the geometrical section, which reads:—

*To a young man at the University.**

Humble work of thy brother, pronounced or to be,
Now rightly appears, devoted to thee ;
Should'st learn from it aught, both happy and sure
In thy patronly favour permit it endure,

and is in the same handwriting as the inscription.

VIII

THE PREFACES

In the following translation of the Prefaces, ordinary type is used instead of Barrow's italics, in order that I may call attention to points already made, or points that will be possibly referred to later, in the notes on the text, by means of italics.

The first Preface, which precedes the Optics :—

"Communication to the reader.

"Worthy reader,

"That this, of whatever humble service it may be, was not designed for you, you will soon understand from many indications, if you will only deign to examine it ; nor, that you might yet demand it as your due, were other authorities absent. To these at least, truly quaking in mind and after great hesitation, I yielded ; chiefly because thereby I should set as an example to my successors the production of a literary work as a duty, such as I myself was the first to discharge ; if less by the execution thereof, at any rate by the endeavour at advance, not unseemly, nor did it seem to be an ostentation foreign, to my office. There was in addition some slight hope that there might be therein something of the nature of good fruit, such as in some measure might profit you, and not altogether be displeasing to you. Also, remember, I warn those of you, who are more advanced in the subject of my book, what manner of writing you are handling ; not elaborated in any way for you alone ;

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* With apologies for doggerel ; but the translation is fairly close, line for line.

not produced on my own initiative; nor by long meditation, exhibiting the ordered concepts of leisurely thought; but Scholastic Lectures; first extracted from me by the *necessities of my office*; then from time to time expanded over-hastily to complete my task within the allotted time; lastly, prepared for the instruction of a promiscuous literary public, for whom it was important not to leave out many lighter matters (as they will appear to you). In this way you will not be looking in vain (and it is necessary to warn you of this, lest by expecting too much you may harm both yourself and me) for anything elaborated, skilfully arranged, or neatly set in order. For indeed I know that, to make the matter satisfactory to you, it would be expedient to cut out many things, to substitute many things, to transpose many things, and to 'recall all to the anvil and file.' For this, however, I had neither the stomach nor the leisure to take the pains; nor indeed had I the capability to carry the matter through. And so I chose rather to send them forth 'in Nature's garb,' as they say, and just as they were born; rather than, by laboriously licking them into another shape, to fashion them to please. However, after that I had entered on the intention of publication, either seized with disgust, or avoiding the trouble to be undergone in making the necessary alterations, in order that I should not indeed put off the rewriting of the greater part of these things, *as delicate mothers are wont, I committed to the foster care of friends, not unwillingly, my discarded child, to be led out and set forth as it might seem good to them.* Of which, for I think it right that you should know them by name, Isaac Newton, a fellow of our college (a man of exceptional ability and remarkable skill) has revised the copy, warning me of many things to be corrected, and adding some things from his own work, which you will see annexed with praise here and there. The other (whom not undeservedly I will call the Mersenne of our race, born to carry through such essays as this, both of his own work and that of others) John Collins has attended to the publication, at much trouble to himself.

"I could now place other obstacles to your expectation, or show further causes for your indulgence (such as my meagre ability, a lack of experiments, other cares intervening) if

I were not afraid that that bit of wit of the elder Cato would be hurled at me:—

"'Truly you publish abroad these things as if bound by a decree of the Amphictyones.'

"At least fairness demanded a prologue of this kind, and in some degree a certain parental affection for one's own offspring enticed it forth, in order that it might stand forth the more excusable, and more defended from censure.

"But if you are severe, and will not admit these excuses into a propitious ear, according to your inclination (I do not mind) you may reprove as much and as vigorously as you please."

The second Preface, which refers to the Geometry:—

"My dear reader,

"Of these lectures (*which you will now receive in a certain measure late-born*), seven (one being excepted) I intended as the final accompaniments and as it were the things left over from the Optical lectures, *which stand forth lately published*; otherwise, I imagine, I shall be thought little of for bringing out sweepings of this kind. However, when the publisher [or editor—*Librarius*—? Collins] thought, for reasons of his own, that these matters should be prepared, separately removed from the others; and moreover he desired something else to be furnished that should give the work a distinct quality of its own (so that indeed it might surpass the size of a supplementary pamphlet); falling in with his wishes (*I will not say very willingly*) *I added the first five lectures*, cognate in matter with those following and as it were coherent; which indeed *I had devised some years ago*, but, as with no idea of publishing, so without that care which such an intention calls for. For they are clumsily and confusedly written; nor do they contain anything firmly, or anything lying beyond the use or the comprehension of the beginners for which they are adapted; wherefore I warn those experienced in this subject to keep their eyes turned *away from these sections*, or at least to give them indulgence a little liberally.

"The other seven, that I spoke of, *I expose more freely to your view*, *hoping that there is nothing in them that it will displease the more erudite to see.*

"The last lecture of all a friend (truly an excellent man, one of the very best, but in a business of this sort an insatiable dun*) extorted from me; or, more correctly, claimed its insertion as a right that was deserved.

"For the rest, *what these lectures bring forth*, *or to what they may lead*, you may easily learn by tasting the beginnings of each.

"Since there is now no reason why I should longer detain or delay you,

"FAREWELL."

IX

HOW BARROW MADE HIS CONSTRUCTIONS

In hazarding a guess as to how Barrow came by his constructions, one has, to a great extent, to be guided by his other works, together with any hint that may be obtained from the *order* of his theorems in the text. Taking the latter first, I will state the effect the reading of the text had on me. The only thing noticeable, to begin with, was the pairing of the propositions, rectangular and polar; the rest seemed more or less a haphazard grouping, in which one proposition did occasionally lead to another; but certain of the more difficult constructions were apparently without any hint from the preceding propositions. Once, however, it began to dawn on me that Barrow was trying to write a *complete elementary treatise on the calculus*, the matter was set in a new light. First, the preparation for the idea of a small part of the tangent being substituted for a small part of the arc, and *vice versa* (Lect. V, § 6), this, of course, having been added later, probably, I suggest, to put the differential triangle on a sound basis; then the lemmas on hyperbolas, for the equivalent of a first approximation in the form of $y = (ax + b)/(cx + d)$ for any equation in the form giving y as an explicit function of x; this first gave the clue pointing to his constructions having been found out analytically; then the work on arithmetical and geometrical means leading to the approximation to the binomial raised to a fractional power; lastly, a few tentative standard forms; and then Lect. IX, with the differentiation of a fractional power, and the whole design is clear as day. Barrow knows the calculus algebraically and is setting it in geometrical form to furnish a rigorous demonstration. From this point onwards, truly with many a sidestep as something especially pretty strikes him as he goes, he proceeds methodically to accumulate the usual collection of standard forms and standard rules for their completion as a calculus. If one judges from this alone, there is no other possible explanation of the plan of the work.

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* *Flagitator improbus;* a specimen of Barrovian humour.

I then looked round for some hint that might corroborate this opinion, and I found it, to me as clear as daylight, in his lectures on the explanation of the method of Archimedes. In these I am convinced *Barrow is telling the story of his own method*, as well as stating the source from which he has derived the idea of such a procedure. With this compare Newton's anagram and Fermat's discreet statement of the manner in which he proved that any prime of the form $4n+1$ was the sum of two squares. Any reader, who has been led, by reading this statement, into trying to produce a proof of this theorem for himself, will agree with me that Fermat was not giving very much of his method away. And so it was with all these mathematicians, and other scientists as well; they stated their results freely enough, and sometimes gave proofs, but generally in a form that did not reveal their own particular methods of arriving at them. For instance, take the construction of Lect. IX, § 10; to my mind there cannot possibly be any doubt that he arrived at it analytically; and the analytical equivalent of it as it stands is

$$\text{If} \qquad v = f(x), \quad \text{and} \quad \mathsf{M}z = \mathsf{N}y + (\mathsf{M}-\mathsf{N})(mx+c),$$
$$\text{then} \qquad \mathsf{M}\,dz/dx = \mathsf{N}\,dy/dx + (\mathsf{M}-\mathsf{N})m;$$

given the capacity for doing this bit of differentiation, the construction given would be easily found by Barrow. This construction is all the more remarkable because the proof given is unsound, not to say wrong; and I suggest that this fact is a very strong piece of evidence that the construction was not arrived at geometrically. Many other examples might be cited, but this one should be sufficient.

X

ANALYTICAL EQUIVALENTS TO BARROW'S CHIEF THEOREMS

Fundamental Theorem

If n is any positive rational number, integral or fractional, then $(1+x)^n \gtrless 1+n.x$, according as $n \gtrless 1$; and this inequality tends to become an equality when x tends to zero.

[Proved without convergence in Lect. VII, §§ 13–16.]

Standard Forms for Differentiation

1. If y is any function of x, and $z = \mathrm{A}/y$,
then $dz/dx = -(\mathrm{A}/y^2).dy/dx$ . . . Lect. VIII, 9

2. If y is a function of x, and $z = y + \mathrm{C}$,
then $dz/dx = dy/dx$ . . . . . Lect. VIII, 11

3. If y is any function of x, and $z^2 = y^2 - a^2$,
then $z.dz/dx = y.dy/dx$; or, in another form,
if $z = \surd(y^2 - a^2)$, then $dz/dx = [y/\surd(y^2 - a^2)]dy/dx$
Lect. VIII, 13

4. If $z^2 = y^2 + a^2$, then $z.dz/dx = y.dy/dx$,
or $dz/dx = [y/\surd(y^2 + a^2)]dy/dx$ . . . Lect. VIII, 14

5. If $z^2 = a^2 - y^2$, then $z.dz/dx = -y.dy/dx$,
or $dz/dx = -[y/\surd(a^2 - y^2)]dy/dx$ . . . Lect. VIII, 15

6. If y is any function of x, and $z = a + by$,
then $dz/dx = b.dy/dx$ . . . . . Lect. IX, 1

7. If y is any function of x, and $z^n = a^{n-r}.y^r$,
then $(1/z).dz/dx = (n/r).(1/y).dy/dx$ . . Lect. IX, 3

8. Special case: $d(x^n)/dx = n.x^{n-1}$ or $n.(y/x)$,
where n is a positive rational . . . Lect. IX, 4

9. The case when n is negative is to be deduced from the combination of Forms 1 and 8.

10. If $y = \tan x$, then $dy/dx = \sec^2 x$, proved as Ex. 5 on the "differential triangle" at the end of Lect. X.

11. It is to be noted that the same two figures, as used for $\tan x$, can be used to obtain the differential coefficients of the other circular functions.

Laws for Differentiation

Law 1. **Sum of Two Functions.**—If $w = y + z$, then $dw/dx = dy/dx + dz/dx$ . . . . Lect. VIII, 5

Law 2. **Product of Two Functions.**—If $w = yz$, then $(1/w).dw/dx = (1/y).dy/dx + (1/z).dz/dx$. Lect. IX, 12

Law 3. **Quotient of Two Functions.**—If $w = y/z$, then, if $v = 1/z$, $(1/v).dv/dx = -(1/z).dz/dx$, as has already been obtained in Lect. VIII, 9; hence by the above—

$$(1/w).dw/dx = (1/y).dy/dx - (1/z).dz/dx.$$

N.B.—Note the logarithmic form of these two results, corresponding with the subtangents used by Barrow.

The remaining standard forms Barrow is unable apparently to obtain directly; and the same remark applies to the rest of the laws. So he proceeds to show that

Differentiation and Integration are inverse operations.

(i.) If R.$z = \int y dx$, then R.$dz/dx = y$ . . Lect. IX, 11

(ii.) If R.$dz/dx = y$, then R.$z = \int y dx$ . . Lect. XI, 19

Hence the standard forms for integration are to be obtained immediately from those already found for differentiation. Barrow, however, proves the integration formula for an integral power independently, in the course of certain theorems in Lect. XI. He also gives a separate proof of the quotient law in the form of an integration, in Lect. XI, 27.

Further Standard Forms for Integration

$$\left.\begin{array}{l}\text{A. } \int_1^x dx/x = log\, x \\ \text{B. } \int_0^x a^x dx = k(a^x - 1)\end{array}\right\} \;.\;\;.\;\;.\; \text{Lect. XII, App. 3, Prob. 3, 4}$$

C. $\int_0^\theta tan\,\theta\, d\theta = log\,(cos\,\theta)$ . . Lect. XII, App. I, 2

D. $\int_0^\theta sec\,\theta\, d\theta = \frac{1}{2}\, log\{(1 + sin\,\theta)/(1 - sin\,\theta)\}$,, ,, 5

E. $\int dx/(a^2 - x^2) = \{log\,(a + x)/(a - x)\}/2a$ (see Form D)

F. $\int cos\,\theta\, d(tan\,\theta)\, d\theta = \int tan\,\theta\, d(cos\,\theta)\, d\theta - tan\,\theta\, cos\,\theta$,

both being equal to $\int sec\,\theta\, d\theta$, the only example of "integration by parts" I have noticed . Lect. XII, App. I, 8

G. $\int dx/\sqrt{(x^2 + a^2)} = log[\{x + \sqrt{(x^2 + a^2)}\}/a]$,, ,, 9

Graphical Integration of any Function

For any function, $f(x)$, that cannot be integrated by the foregoing rules, Barrow gives a graphical method for $\int f(x)dx$ as a logarithm of the quotient of two radii vectores of the curve $r=f(\theta)$, and for $\int dx/f(x)$ as a difference of their reciprocals . . . Lect. XII, App. III, 5–8

Fundamental Theorem in Rectification

He proves that $(ds/dx)^2 = 1 + (dy/dx)^2$ . . Lect. X, 5

He rectifies the cycloid (thus apparently anticipating Wren), the logarithmic spiral, and the three-cusped hypocycloid (as special cases of one of his general theorems), and reduces the rectification of the parabola to the quadrature of the rectangular hyberbola, from which the rectification follows at once.

(XII, App. III, 1, Ex. 2; XI, 26; XII, 20, Ex. 3.)

In addition to the foregoing theorems in the Infinitesimal Calculus (for if it is not a treatise on the elements of the Calculus, what is it?), Barrow gives the following interesting theorems in the appendix to Lect. XI.:—

Maxima and Minima

He obtains the maximum value of $x^r(c-x)^s$, giving the condition that $x/r=(c-x)/s$; also he shows that this is the condition for the minimum value of $x^r/(x-c)^s$.

Trigonometrical Approximations

Barrow proves that the circular measure of an angle α lies between $3 \sin\alpha/(2+\cos\alpha)$ and $\sin\alpha(2+\cos\alpha)/(1+2\cos\alpha)$, the former being a lower limit, and equivalent to the formula of Snellius; each of these approximations has an error of the order of α^5.

THE
GEOMETRICAL LECTURES

ABRIDGED TRANSLATION

WITH NOTES, DEDUCTIONS, PROOFS OMITTED
BY BARROW, AND FURTHER EXAMPLES OF
HIS METHOD

LECTURE I

Generation of magnitudes. Modes of motion and the quantity of the motive force. Time as the independent variable. Time, as an aggregate of instants, compared with a line, as the aggregate of points. Deductions.

[In this lecture, Barrow starts his subject with what he calls the generation of magnitudes.]

Every magnitude can be either supposed to be produced, or in reality can be produced, in innumerable ways. The most important method is that of "local movements." In motion, the matters chiefly to be considered are the mode of motion and the quantity of the motive force. Since quantity of motion cannot be discerned without Time, it is necessary first to discuss Time. Time denotes not an actual existence, but a certain capacity or possibility for a continuity of existence; just as space denotes a capacity for intervening length. Time does not imply motion, as far as its absolute and intrinsic nature is concerned; not any more than it implies rest; whether things move or are still, whether we sleep or wake, Time pursues the even tenor of its way. Time implies motion to be measurable; without motion we could not perceive the passage of Time.

"*On Time, as Time, 'tis yet confessed*
From moving things distinct, or tranquil rest,
No thought can be,"

is not a bad saying of Lucretius. Also Aristotle says:—"*When we, of ourselves, in no way alter the train of our thought, or indeed if we fail to notice things that are affecting it, time does not seem to us to have passed.*" And indeed it does not appear that any, nor is it apparent how much, time has elapsed, when we awake from sleep. But from this, it is not right to conclude that:—"*It is plain that Time does not exist without motion and change of position.*" "*We do not perceive it, therefore it does not exist,*" is a fallacious inference; and sleep is deceptive, in that it made us connect two widely separated instants of time. However, it is very true that:—"*Whatever the amount of the motion was, so much time seems to have passed*"; nor, when we speak of so much time, do we mean anything else than that so much motion could have gone on in between, and we imagine the continuity of things to have coextended with its continuously successive extension.

We evidently must regard Time as passing with a steady flow; therefore it must be compared with some handy steady motion, such as the motion of the stars, and especially of the Sun and the Moon; such a comparison is generally accepted, and was born adapted for the purpose by the Divine design of God (Genesis i, 14). But how, you say, do we know that the Sun is carried by an equal motion, and that one day, for example, or one year, is exactly equal to another, or of equal duration? I reply that, if the sun-dial is found to agree with motions of any kind of time-measuring instrument, designed to be moved uniformly by successive repetitions of its own peculiar motion, under suitable conditions, for whole periods or for proportional parts of them; then it is right to say that it registers an equable motion. It seems to follow that strictly speaking the celestial bodies are not the first and original measures of Time; but rather those motions, which are observed round about us by the senses and which underlie our experiments, since we judge the regularity of the celestial motions by the help of these. On the other hand, Time may be used as a measure of motion; just as we measure space from some magnitude, and then use this space to estimate other magnitudes commensurable with the first; *i.e.* we compare motions with one another by the use of time as an intermediary.

Time has many analogies with a line, either straight or circular, and therefore may be conveniently represented by it; for time has length alone, is similar in all its parts, and can be looked upon as constituted from a simple addition of successive instants or as from a continuous flow of one instant; either a straight or a circular line has length alone, is similar in all its parts, and can be looked upon as being made up of an infinite number of points or as the trace of a moving point.

Quantity of the motive force can similarly be thought of as aggregated from indefinitely small parts, and similarly represented by a straight line or a circular line; when Time is represented by a distance the motive force is the same as the velocity. Quantity of velocity cannot be found from the quantity of the space traversed only, nor from the time taken only, but from both of these brought into reckoning together; and quantity of time elapsed is not determined without known quantities of space and velocity; nor is quantity of space (so far as it may be found by this method) dependent on a definite quantity of velocity alone, nor on so much given time alone, but on the joint ratio of both.

To every instant of time, or indefinitely small particle of time, (I say instant or indefinite particle, for it makes no difference whether we suppose a line to be composed of points or of indefinitely small linelets; and so in the same manner, whether we suppose time to be made up of instants or indefinitely minute timelets); to every instant of time, I say, there corresponds some degree of velocity, which the moving body is considered to possess at the instant; to this degree of velocity there corresponds some length of space described (for here the moving body is a point, and so we consider the space as merely long). But since, as far as this matter is concerned, instants of time in nowise depend on one another, it is possible to suppose that the moving body in the next instant admits of another degree of velocity (either equal to the first or differing from it in some proportion), to which therefore will correspond another length of space, bearing the same ratio to the former as the latter velocity bears to the preceding; for we cannot but suppose that our instants are exactly equal to one another. Hence, if to every instant of time there is assigned a suitable degree of velocity, there will be aggregated out of these a certain quantity, to any parts of which respective parts of space traversed will be truly proportionate; and thus a magnitude representing a quantity composed of these degrees can also represent the space described. Hence, if through all points of a line representing time are drawn straight lines so disposed that no one coincides with another (*i.e.* parallel lines), the plane surface that results as the aggregate of the parallel straight lines, when each represents the degree of velocity corresponding to the point through which it is drawn, exactly corresponds to the aggregate of the degrees of velocity, and thus most conveniently can be adapted to represent the space traversed also. Indeed this surface, for the sake of brevity, will in future be called the aggregate of the velocity or the representative of the space. It may be contended that rightly to represent each separate degree of velocity retained during any timelet, a very narrow rectangle ought to be substituted for the right line and applied to the given interval of time. *Quite so, but it comes to the same thing whichever way you take it*; but as our method seems to be simpler and clearer, we will in future adhere to it.

[Barrow then ends the lecture with examples, from which he obtains the properties of uniform and uniformly accelerated motion.]

(1) If the velocity is always the same, it is quite evident from what has been said that the aggregate of the velocity attained in any definite time is correctly represented by a parallelogram, such as AZZE, where the side AE stands for

a definite time, the other AZ, and all the parallels to it, BZ, CZ, DZ, EZ, separate degrees of velocity corresponding to the separate instants of time, and in this case plainly equal to one another. Also the parallelograms AZZB, AZZC, AZZD, AZZE, conveniently represent, as has been said, the spaces described in the respective times, AB, AC, AD, AE.

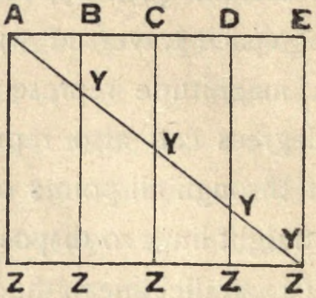

Fig. 1.

(2) If the velocity increase uniformly from rest, then the aggregate of the velocities is represented by the triangle AEY. Also if the velocity increases uniformly from some definite velocity to another definite velocity represented respectively by CY, EY, then the space is represented by a trapezium, such as CYYE.

(3) If the velocity increase according to a progression of square numbers, the space described to represent the aggregate of the velocity is the complement of a semi-parabola. [For which Barrow gives a figure.]

[From (1) and (2) all the properties of uniform motion and of uniformly accelerated motion are simply deduced, and the lecture concludes with the remark :—]

These things, being necessary for the understanding of things to be said later, and theories of motion that are, I think, not on the whole quite useless, it has seemed to be advantageous to explain clearly as a preliminary. Having finished this task, I direct my steps forward.

NOTE

There is not much in this lecture calling for remark. The matter, as Barrow says in his Preface, is intended for beginners. There is, however, the point, to which attention is called by the italics on page 39, that Barrow fails to see any difference between the use of lines and narrow rectangles as constituent parts of an area. This is Cavalieri's method of "indivisibles," which Pascal showed incontrovertibly was the same as the method of "exhaustions," as used by the ancients. There is evidence in later lectures that Barrow recognized this; for he alludes to the possibility of an alternative indirect argument (*discursus apogogicus*) to one of his theorems, and later still shows his meaning to be the method of obtaining an upper and a lower limit. There is also a suggestion that he personally used the general modern method of the text-books, that of proving that the error is less than a rectangle of which one side represented an instant and the other the difference between the initial and final velocities; and that it could be made evanescent by taking the number of parts, into which the whole time was divided, large enough. Also the attention of those who still fight shy of graphical proofs for the laws of uniformly accelerated motion, if any such there be to-day, is called to the fact that these proofs were given by such a stickler for rigour as Barrow, with the remark that they are evident, at a glance, from the diagrams he draws.

LECTURE II

Generation of magnitudes by "local movements." The simple motions of translation and rotation.

Mathematicians are not limited to the actual manner in which a magnitude has been produced; they assume any method of generation that may be best suited to their purpose.* Magnitudes may be generated either by simple motions, or by composition of motions, or by concurrence of motions.

[Examples of the difference of meaning that Barrow attaches to the two latter phrases are given by him in a later lecture. The simple motions are considered in this lecture.]

There are two kinds of simple motions, translation and rotation, *i.e.* progressive motion and motion in a circle. For these motions, mathematicians assume that (1) a point can progress straightforwardly from any fixed terminus, and describe a straight line of any length; (2) a straight line can proceed with one extremity moving along any other line, keeping parallel to itself; the former is called the *genetrix*, and is said to be *applied* to the latter which is called the *directrix*; by these are described parallelogrammatic surfaces, when the genetrix and the directrix are both in the same plane, and prismatic and cylindrical surfaces otherwise. In general, the genetrix may, if necessary, be taken as a curve, which is intended to include polygons, and the genetrix and the directrix may usually be interchanged. The same kinds of assumptions are made for simple motions of rotation; and by these are described circles and rings and sectors or parts of these, when a straight line rotates in its own plane about a point in itself or in the line produced; if the directrix is a curve (in the wider sense given above), and does not lie in the plane of the genetrix, of which one point is supposed to be fixed, the surfaces generated are pyramidal or conical. From this kind of generation is deduced the similarity of parallel sections of such surfaces; and thus it is evident that the surfaces can also be generated by taking the genetrix of the first method as the directrix, and the former directrix as the genetrix so long as it is supposed to shrink proportionately as it proceeds parallel to itself towards what was the vertex or fixed point in the first method.

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* As an example, take the case of finding the volume of a right circular cone by integration; here, by definition, the method of generation is the rotation of a right-angled triangle about one of the rectangular sides; but it is supposed to be generated, for the purpose of modern integration, by the motion of a circle, that constantly increases in size, and moves parallel to itself with its centre on the axis of the cone.

For producing solids the chief method is a simple rotation, about some fixed line as axis, of another line lying in the same plane with it. In addition, there is the method of "indivisibles," which in most cases is perhaps the most expeditious of all, and not the least certain and infallible of the whole set.

The learned A. Tacquetus * more than once objects to this method in his clever little book on "Cylindrical and Annular Solids," and therein thinks that he has falsified it, because the things found by means of it concerning the surfaces of cones and spheres (I mean quantities of these) do not agree in measurement with the truths discovered and handed down by Archimedes.

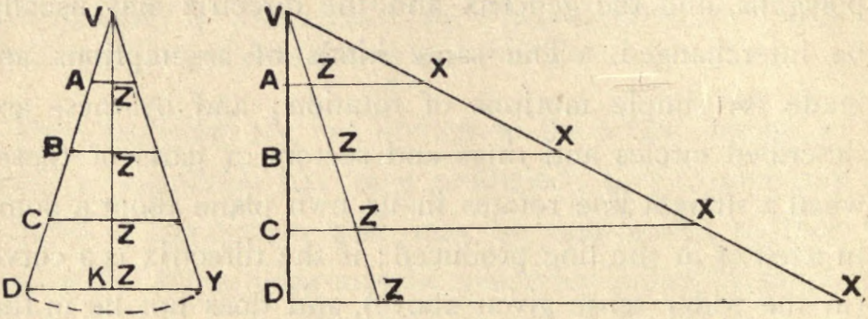

Fig. 10.†

Take, for example, a right cone DVY, whose axis is VK; through every point of this suppose that there pass straight lines ZA, ZB, ZC, ZD (or KD), etc.; from these indeed according to the Atomic theory the right-angled triangle VDK is made up; and from the circles described with these as radii the cone itself is made. "*Therefore*," he argues, "*from the peripheries of these circles is composed the conical surface; now this is found to be contrary to the truth; hence the method is fallacious.*"

I reply that the calculation is wrongly made in this manner; and in the computation of the peripheries of which such a surface is composed, a reasoning has to be adopted different from that used when computing the lines from which plane surfaces are made up, or the planes from which solids are formed. In fact, it must be considered that the multitude of peripheries forming the curved surface are produced, through the rotation of the line VD, *from the multitude of points in the genetrix* VD *itself*; by observing this distinction all error will be obviated, as I will now demonstrate.*

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* Andreas Tacquet, a Jesuit of Antwerp: published a book on the cylinder (1651), *Elementa Geometriæ* (1654) and a book on Arithmetic (1664); mentioned by Wallis.

† The numbering of the diagrams is Barrow's and is, in consequence of abridgment, not consecutive.

At every point of the line VD, instead of the line VK, suppose that right lines are applied perpendicular to the line VD, and equal to the peripheries, taken in order, that make up the curved surface. From these parallel straight lines is generated the plane VDX, which is equal to the said curved surface.

Further, if instead of the peripheries we apply the corresponding radii, the space produced will bear to the curved surface a ratio equal to that of the radius of any circle to its circumference. In the particular case chosen, the two

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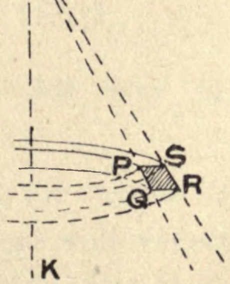
* This is the first example we come across of the superiority of Barrow's insight into what is really the method of integration. In effect, Barrow points out that if a periphery is thought of as a solid ring of very minute section, then in this case the section is a trapezium, as shown in the annexed diagram, of which the parallel sides are perpendicular to the axis of the cone, and the non-parallel sides both pass, if produced, through the vertex. Tacquet uses the surface generated by the top parallel sides, PS as if he were finding the area of a circle, by means of concentric rings (? or he uses the perpendicular distance from S on QR); Barrow points out that he should use the surface generated by the slant side SR.

In modern phraseology, Barrow shows that Tacquet has made the error of integrating along a radius of the base (? or along the axis), instead of along a slant side.



plane surfaces are triangles and the area of the curved surface is thus easily found.

There are other methods which may be used conveniently in certain cases; but enough has been said for the present concerning the construction of magnitudes by simple motions.

NOTE

It would be interesting to see how Barrow would get over the difficulty raised by Tacquet, if Tacquet's example had been the case of the oblique circular cone. It seems to me to be fortunate for Barrow that this was not so. Barrow also states, be it noted, that the method is general for any solid of revolution, if the generating line is supposed to be straightened before the peripheries are applied; in which case, the area can be found for the curved surface only when the plane surface aggregated from the applied peripheries turns out to be one whose dimensions can be found.

Thus, if the ordinate varies as the square of the arc measured from the vertex, the plane equivalent is a semi-parabola, and the area is $2\pi sr/3$, where s is length of the rotating arc, and r is the maximum or end ordinate.

LECTURE III

Composite and concurrent motions. Composition of rectilinear and parallel motions.

In generation by composite motions, if the remaining motions are unaltered, then, according as the velocity of one, or more, is altered, we usually obtain magnitudes differing not only in kind but also in quantity, or at least differing in position every time.

Thus, suppose the straight line AB is carried along the straight line AC by a uniform parallel motion, and at the same time a point M descends uniformly in AB; or suppose that, while AC descends with a uniform parallel motion, it cuts AB also moving uniformly and to the right. From motions of this kind, *composite* in the former case, and *concurrent* in the latter, the straight line AM may be produced.

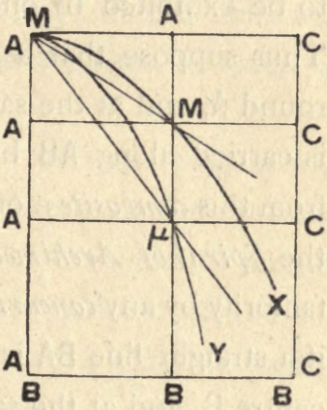


Fig. 13.

Again, if in the previous example, whilst the motion of the straight line AB remains the same as before with respect to its velocity, but the uniform motion of the point M, or of the straight line AC, is altered in velocity, so that indeed the point M now comes to the point μ, or AC cuts AB in μ, there is described by this motion another straight line Aμ, in a different position from the first.

Further, if once more, while the motion of AB remains the same, instead of the uniform motion of the point M, or of the straight line AC, we substitute a motion that is called uniformly accelerated; from such composite or concurrent motion is produced the parabolic line AMX, or in another case the line AμY, according as the accelerated motion is supposed to be one thing or another in degree.

In these examples, it is seen that composite and concurrent motions come to the same thing in the end; but frequently the generation of magnitudes is not so easily to be exhibited by one of these methods as by the other. Thus suppose that a straight line AB is uniformly rotated round A, and at the same time the point M, starting from A, is carried along AB by a continuous and uniform motion; from this *composite* motion is produced a certain line, namely the *Spiral of Archimedes*, which cannot be explained satisfactorily by any *concurrence* of motions. On the other hand, if a straight line BA is rotated with uniform motion about a centre B, and at the same time a straight line AC is moved in a parallelwise manner uniformly along AB, the continuous intersections of BA, AC, so moving, form a certain line, usually called the *Quadratrix*; and the generation of this line is not so clearly shown or explained by any strictly so-called composition of motion.

Magnitudes can be compounded and also decomposed in innumerable ways; but it is impossible to take account of all of these, so we shall only discuss some important cases, such as are considered to be of most service and the more easily explained. Such especially are those that are compounded of rectilinear and parallel motions, or rectilinear and rotary motions, or of several rotary motions; preference being given to those in which the constituent simple motions are all, or at least some of them, uniform. Moreover, there is not any magnitude that cannot be considered to have been generated by rectilinear motions alone. For every line that lies in a plane can be generated by the motion of a straight line parallel to itself, and the motion of a point along it; every surface by the motion of a plane parallel to itself and the motion of a line in it (that is, any line on a curved surface can be generated by rectilinear motions); in the same way solids, which are generated by surfaces, can be made to depend on rectilinear motions.

I will only consider the generation of lines lying in one plane by rectilinear and parallel motions; for indeed there is not one that cannot be produced by the parallel motion of a straight line together with that of a point carried along it;* but the motions must be combined together as the special nature of the line demands.

For instance, suppose that a straight line ZA is always moved along the straight line AY parallelwise, by any motion, uniform or variable, increasing or decreasing or alternating in velocity, according to any imaginable ratio; and that any point M in it is moved in such a way that the motion of the point is proportional to the motion of the straight line, throughout any the same intervals of time; then there will be certainly a straight line generated by these motions.

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* In other words, Barrow states that every plane curve has a Cartesian equation, referred to either oblique or rectangular coordinates; yet it is doubtful whether he fully recognizes that *all* the properties of the curve can be obtained from the equation.

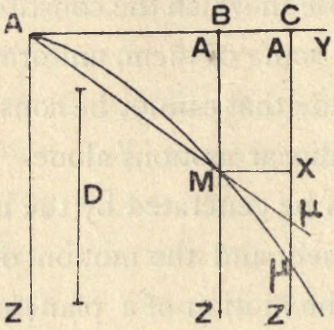


Fig. 17.

For, since we always have

$$AB : AC = BM : C\mu, \quad \text{or} \quad AB : MX = AM : X\mu,$$

(MX being drawn parallel to AC), it follows that the points A, M, μ are in one straight line.

But if, instead, these motions are comparable with one another in such fashion that, given any line D, then the rectangle, contained by the difference between the line D and BM the distance traversed and BM itself, always bears the same ratio to the square on AB (the distance traversed by the line AZ in the same time); then a circle or an ellipse is described; a circle, if the supposed ratio is one of equality and the angle ZAY is a right angle, and an ellipse otherwise; and in these there will be one diameter, situated in the line AZ in its first position, and drawn from A in the direction of Z, and this diameter will be equal to D.

If, however, the motions are such that the rectangle contained by the sum of the lines D and BM and BM itself bears always the same ratio to the square on the line AB, a hyperbola will be produced; a rectangular or equilateral hyperbola, if the assigned ratio is one of equality and the angle ZAY is a right angle; if otherwise, of another kind, according to the quality of the assigned ratio; of these the transverse diameter will be equal to D, being situated in ZA when occupying its first position, and being measured in a direction opposite to Z; and the parameter is given by the given ratio.

But if the rectangle contained by D and BM bears always the same ratio to the square on AB, it is evident that a parabolic line is produced, of which the parameter is easily found from the given straight line D and the quantity of the assigned ratio.

Also in the first case of these, if the transverse motion along AY is supposed to be uniform, the descending motion along AZ will also be uniform; in the second and third cases, if the motion along AY is uniform, the descending motion along AZ will be continually increasing; and the same thing being supposed in the last case, in which the parabola is produced, the point M has its velocity increased uniformly.

In a similar manner, any other line can be conceived to be produced by such a composition of motion. But we shall come across these some time or other as we go along; let us see whether anything useful in mathematics can be obtained from a supposed generation of lines in this way.

But for the sake of simplicity and clearness let us suppose that one of the two motions, say that of the line preserving parallelism, is always uniform; and let us strive to make out what general properties of the generated lines arise from the general differences with regard to the velocity of the other; let us try, I say, but in the next lecture.

Note

We here see the reason for Barrow considering time as the independent variable; he states, indeed, that the constructions can be effected, no matter what is the motion of the line preserving parallelism; but for the sake of simplicity and clearness he decides to take this motion as uniform; for this the consideration of time is necessary. At the same time it is to be noted that Barrow, except for the first case of the straight line, is unable to explicitly describe the velocity of the point M, and uses a geometrical condition as the law of the locus; in other words, he gives the pure geometry equivalent of the Cartesian equation. In later lectures, we shall find that he still further neglects the use of time as the independent variable. This, as has been explained already, is due to the fact that the first five lectures were added *afterwards*. In Barrow's original design, the independent variable is a length along one of his axes. This length is, it is true, divided into equal parts; but that is the only way, a subsidiary one, in which time enters his investigations; and even so, the modern idea of "steps" along a line, used in teaching beginners, is a better analogue to Barrow's method that that which is given by a comparison with fluxions.

LECTURE IV

Properties of curves arising from composition of motions. The gradient of the tangent. Generalization of a problem of Galileo. Case of infinite velocity.

Hereafter, for the sake of brevity, I shall call a parallel motion of the straight line AZ along AY a "transverse motion," and the motion of a point moving from A in the line AZ a "descent" or a "descending motion," regard being had of course to the given figure. Also I shall take the motion along AY and parallels to it as uniform, hence this motion and parts of it can represent the time and parts of the time. Now I come to the properties of lines produced by a uniform transverse motion and a continually increasing descent.

1. The line produced is curved in all its parts.

2. The velocity of the uniform descending motion, by which a curve is described (*i.e.* if there is a common uniform transverse motion for the chord and its arc) is less than the velocity, which the increasing descending motion has at N, the common end of both.

3. Of a curve of this sort, any chord, as MO, falls entirely

within the arc, and if produced, falls entirely without the curve.

This property was separately proved for the circle by Euclid, for the conic sections by Apollonius, for cylinders by Serenus.

4. Curves of this sort are convex or concave to the same parts throughout.

This is the same as saying that a straight line only cuts the curve in two points; nor does it differ from the definition of "hollow," as given by Archimedes at the beginning of his book on the sphere and the cylinder.

5. All straight lines parallel to the genetrix cut the curve; and any one cuts the curve in one point only.

This was proved, separately, for the parabola and the hyperbola by Apollonius, and for the sections of the concoids by Archimedes.

6. Similarly all parallels to the directrix cut the curve, and in one point only.

Apollonius proved this for the sections of the cone.

7. All chords of the curve meet the genetrix and all parallels to it, produced if necessary.

Apollonius thought it worth while to prove this property separately for the parabola and the hyperbola.

8. Similarly, all straight lines touching the curve, with one exception (see § 18), meet the same parallels.

This also Apollonius showed for the conic sections in separate theorems.

9. Also any straight lines cutting the genetrix will also cut the curve.

Apollonius went to a very great deal of trouble to prove a property of this kind in the case of the conic sections.

10. Straight lines applied to the directrix, *i.e.* parallels to the genetrix, have a ratio to one another (when the less is the antecedent) which is less than the ratio of the corresponding spaces traversed by the moving straight line; *i.e.* the ratio of the versed sines of the curve, the less to the greater, is less than the ratio of the sines.

This property of circles and other curves, it will be found, is everywhere proved separately for each kind.

NOTE

All the preceding properties are deduced in a very simple manner from one diagram; and Barrow's continual claim that his method not only simplifies but generalizes the work of the early geometers is substantiated.

The full proof of the next property is given, to illustrate Barrow's way of using one of the variants of the method of exhaustions.

11. Let us suppose that a straight line TMS touches a given curve at a point M (*i.e.* it does not cut the curve); and let the tangent meet AZ in T, and through M let PMG be drawn parallel to AY. I may say that the velocity of the descending point, describing the curve by its motion, which it has at the point

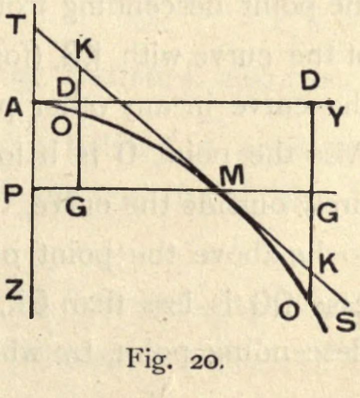
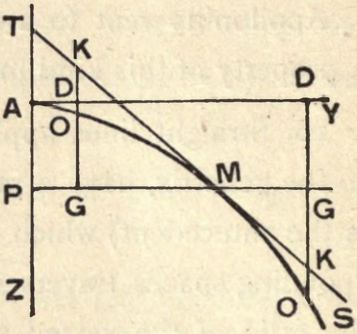


Fig. 20.

of contact M, is equal to the velocity by which the straight line TP would be described uniformly, in the same time as the straight line AZ is carried along AC or PM (or, what comes to the same thing, I say that the velocity of the descending point at M has the same ratio to the velocity with which the straight line AZ is moving as the straight line TP has to the straight line PM).



For, let us take anywhere in the tangent TM any point K, and through it draw the straight line KG, meeting the curve in O and the parallels AY, PG in D, G. Then, since the tangent TM is supposed to be described by a twofold uniform motion, partly of the straight line TZ carried parallelwise along AC or PM, and partly of the point T descending from T along TZ; and since, of these motions, the one along AC or PM is common with, or the same as, that by which the curve is described; and since, when TZ is in the position KG, AZ will be in the same position as well; therefore, when the point descending from T is at K, the point descending from A will be at O, the intersection of the curve with KG (for the straight line KG cannot cut the curve in any other point, as has already been shown). Also the point O is below K, because the tangent lies entirely outside the curve. Now, if the point O is supposed to be above the point of contact, towards T, since in that case OG is less than GK, it is clear that the velocity of the descending point, by which the curve is described, at the point O is less than the velocity of the uniformly descending motion, by which the tangent is produced; since the former, always increasing, in the same time (namely that represented by GM) traverses a smaller space than the latter which does not increase at all; and as this goes on continually, the former describes the straight line OG whilst the latter describes the straight line KG. On the other hand, if the point K is below the point of contact towards the end S, since OG is then greater than KG, it is clear that the velocity of the descending point, by which the curve is produced, at the point O, in the same way as before, is greater than the velocity of the uniformly descending motion, by which the tangent is described; for the former motion, continually decreasing during the time represented by GM, traverses a greater space than the latter, which does not decrease at all, but keeping constant, describes indeed the space KG. Hence, since the velocity of the point describing the curve, at any point of the curve above the point of contact towards A, is less than the velocity of the motion for TP; and at any point of the curve below the point of contact is greater than it; it follows that it is exactly equal to it at the point M.

12. The converse of the preceding theorem is also true.

13. From these two theorems, it follows at once that curves of this kind are touched by any one straight line in one point only.

This, separately, Euclid proved for the circle, Apollonius for the conic sections, and others for other curves.

From this method, then, there comes out an advantage not to be despised, that by the one piece of work propositions are proved concerning tangents in several cases.

14. The velocities of the descending point at any two assigned points of a curve have to one another the ratio reciprocally compounded from the ratios of the lines applied to the straight line AZ from these points (*i.e.* parallels to AY) and the intercepts by the tangents at these points measured from the said applied lines. In other words, the ratio of the velocities is equal to the ratio of the intercepts divided by the ratio of the applied lines.

15. Incidentally, I here give a general solution by my method, and one quite easy to follow, of that problem which Galileo made much of, and on which he spent much trouble, about which Torricelli said that he found it most skilful and ingenious. Torricelli thus enunciates it (for the enunciation of Galileo is not at hand):—

"Given any parabola with vertex A, it is required to find some point above it, from which if a heavy body falls to A, and from A, with the velocity thus attained, is turned horizontally, then the body will describe the parabola."

NOTE

Barrow gives a very easy construction for the point, and a short simple proof; further his construction is perfectly general for any curve of the form $y = x^n$, where n is a positive integer.

16, 17. These are two ingenious methods for determining the ratio of the abscissa to the subtangent.

Barrow remarks that the theorems will be proved more geometrically later, so that they need not be given here.

18. A circle, an ellipse, or any "returning" curve of this kind, being supposed to be generated by this method, then the point describing any one of them must have an infinite velocity at the point of return.

For instance, let a quadrant APM be so generated; then since the tangent, TM, is parallel to the diameter AZ, and only meets it at an infinite distance, therefore the velocity at M is to the velocity of the uniform motion of AZ parallel to itself as an infinite straight line is to PM; hence, the velocity at M must certainly be infinite. And indeed it will be so for all curves of this kind; but for others which are gradually continued to infinity (such as the parabola or the hyperbola) the velocity of the descending point at any point on the curve is finite.

Leaving this, let us go on to those other properties of the given curves which have to be expounded.

NOTE

It is to be observed that, although Barrow usually draws his figures with the applied lines at right angles to his directrix AZ, his proofs equally serve if the applied lines are oblique, in all cases when not otherwise stated. That is, analytically, his axes may be oblique or rectangular. Having mentioned this point, since my purpose is largely with Barrow's work on the gradient of the tangent, I shall always draw the applied lines at right angles, as Barrow does; except in the few isolated cases where Barrow has intentionally drawn them oblique.

LECTURE V

Further properties of curves. Tangents. Curves like the Cycloid. Normals. Maximum and minimum lines.

1. The angles made with the applied lines by the tangents at different points of a curve are unequal; and those are less which are nearer to the point A, the vertex.

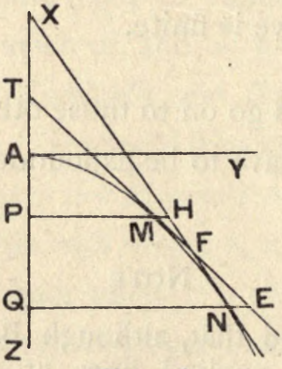


Fig. 26.

2. Hence it may be taken as a general theorem that tangents cut one another between the applied lines drawn at right angles to AZ through the points of contact.

3. The angle PTM is greater than the angle XQN.

4. Applied lines nearer to the vertex (and therefore also any straight lines parallel to other directions) cut the curve at a greater angle than those more remote.

5. If the angle made by an applied line is a right angle or obtuse, I say that the arc MN of the curve is greater than the straight line MN, but less than the straight line ME.

This is a most useful theorem for service in proving properties of tangents. For, it follows from it that, *if the arc* MN *is assumed to be indefinitely small, we may safely substitute instead of it the small bit of the tangent, i.e. either* ME *or* NH.

NOTE

We have here the statement of the fundamental idea of Barrow's method, to which all the preceding matter has led. This is a fine illustration of Barrow's careful treatment; and it is to be observed that this idea is not quite the same thing as the idea of the differential triangle as one is accustomed to consider it nowadays, *i.e.* as a triangle of which the hypotenuse is an infinitely small arc of the curve that may be considered to be a straight line. It will be found later that Barrow uses the idea here given in preference to the other, because by this means the similarity of the infinitesimal triangle with the triangle TPM is far more clearly shown on his diagrams; and many matters in Barrow are made subservient to this endeavour to attain clearness in his diagrams. For instance, when he divides a line into an infinite number of parts, he generally uses four parts on his figure, and gives the demonstration with the warning "on account of the infinite division" as a preliminary statement.

As an example of the use to which the above theorems may be put, Barrow finds the tangent to the Cycloid, his construction being applicable to all curves drawn by the same method. Note that this is not the general case of the roulette discussed by Descartes. Barrow's construction and proof are given in full to bring out the similarity of his criterion of tangency to Fermat's idea, as mentioned in the Introduction.

6. A straight line AY, moving parallel to itself, traverses any curve, either concave or convex to the same parts, with uniform motion (that is to say, it passes over equal parts of the curves in equal times), and simultaneously any point is carried, also uniformly, along AY from A; by the point moving in this manner there is generated a curve AMZ, of which it is required to find the tangent at any point M.

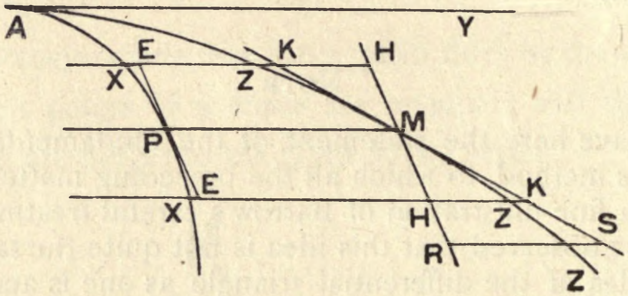


Fig. 27.

To do this, draw MP parallel to AY to cut the curve APX in P; through P draw the straight line PE touching the curve APX; through M draw MH parallel to PE; take any point R in MH, and draw RS parallel to PM; mark off RS so that MR : RS = arc AP : PM (*i.e.* as the one uniform motion is to the other); join MS. Then MS will touch the curve AMZ.

For, if any point Z be taken in this curve, and through it ZK be drawn parallel to MP, cutting the curve APX in X, the tangent at P in E, MH the parallel to it in H, and MS in S; then,

(i), if the point Z is above M towards A, $\quad$ PE < arc PX;

$$\therefore \text{arc } PA : PE > \text{arc } PA : \text{arc } PX.$$

$$\text{But arc } PA : \text{arc } PX = PM : PM - XZ = PM : EH - XZ$$

$$\text{arc } PA : \text{arc } PX = PM : ZH - EX > PM : ZH;$$

$$\text{hence, arc } PA : PE > PM : ZH \quad \text{or} \quad \text{arc } PA : PM > PE : ZH.$$

$$\text{But arc } PA : PM = MR : RS = MH : KH = PE : KH;$$

$$PE : KH > PE : ZH, \quad \text{and} \quad KH < ZH.$$

Now, since $EZ < XZ < PM$ or EH, the point H is outside the curve AZM; hence K is outside the curve AZM.

Similarly [Barrow gives it in full], (ii), if the point Z is below the point M, K will be outside the curve; therefore the whole straight line $KMKS$ lies outside the curve, and thus touches it at M.

After this digression we will return to other properties of the curve.

7. Any parallel to the tangent TM, through a point E directly below T, will meet the curve. [Fig. 26.]

8. If E lies between the point T and the vertex A, the parallel to the tangent will cut the curve twice.

Apollonius was hard put to it to prove these two theorems for the conic sections.

9. If any two lines are equally inclined to the curve, these straight lines diverge outwardly, *i.e.* they will meet one another when produced towards the parts to which the curve is concave.

10. If a straight line is perpendicular to a curve, and along it a definite length HM is taken, then HM is the shortest of all straight lines that can be drawn to the curve from the point H.

11. It follows that the circle, with centre H, drawn through M, touches the curve.

12. Conversely, if HM is the shortest of all straight lines that can be drawn from H to the curve, then HM will be perpendicular to the curve.

13. If HM is the shortest of all straight lines that can be drawn from H, and if the straight line TM is perpendicular to it, then TM touches the curve.

14. Further, a line which is nearer to HM is shorter than one which is more remote.

15. Hence it follows that any circle described with centre H meets the curve in one point only on either side of M; that is, it does not cut the curve in more than two points altogether.

16. If two straight lines are parallel to a perpendicular, the nearer of these will fall more nearly at right angles to the curve than the one more remote.

17. If from any point in the perpendicular HM, two straight lines are drawn to the curve, the nearer will fall more nearly at right angles to the curve than the one more remote.

18. Hence it is evident that by moving away from the perpendicular, the obliquity of the incident lines with the curve increases, until that which touches the curve is reached; this, the tangent, is the most oblique of all.

19. If the point H is taken within the curve, and if, of

all lines drawn from it to meet the curve, HM is the least; then HM will be perpendicular to the curve or the tangent MT.

20. Also, if HM is the greatest of all straight lines drawn to meet the curve, then HM will be perpendicular to the curve.

21. Hence, if MT is perpendicular to HM, whether the latter is a maximum or a minimum, it will touch the curve.

22. It follows that, if a straight line is not perpendicular to the curve, no greatest or least can be taken in it.

23. If HM is the least of the lines drawn to the curve, and any point I is taken in it; then IM will be a minimum.

24. If HM is the greatest of the lines drawn to meet the curve, and any point I is taken on MH produced; then IM will be a maximum.

For the rest, the more detailed determination of the greatest and least lines to a curve depends on the special nature of the curve in question

[Barrow concludes these preliminary five lectures with the remark :—]

"But I must say that it seems to me to be wrong, and not in complete accord with the rules of logic, to ascribe things which are applicable to a whole class, and which come from a common origin, to certain particular cases, or to derive them from a more limited source."

NOTE

The next lecture is the first of the seven, as originally designed, that were to form a supplement to the Optics. Barrow begins thus:—

"I have previously proved a number of general properties of curves of continuous curvature, deducing them from a certain mode of construction common to all; and especially those properties, as I mentioned, that had been proved by the Ancient Geometers for the special curves which they investigated. Now it seems that I shall not be displeasing, if I shall add to them several others (more abstruse indeed, but not altogether uninteresting or useless); these will be, as usual, demonstrated as concisely as possible, yet by the same reasoning as before; this method seems to be in the highest degree scientific, for it not only brings out the truth of the conclusions, but opens the springs from which they arise.

The matters we are going to consider are chiefly concerned with

(i) An investigation of tangents, *freed from the loathsome burden of calculation*, adapted alike for investigation and proof (by deducing the more complex and less easily seen from the more simple and well known);

(ii) The ready determination of the dimensions of many magnitudes by the help of tangents which have been drawn.

These matters seem not only to be somewhat difficult compared with other parts of Geometry, but also they have not been as yet wholly taken up and exhaustively treated (as the other parts have); at the least *they have not as yet been considered according to this method that I know*. So we will straightway tackle the subject, proving as a preliminary certain lemmas, which we shall see will be of considerable use in demonstrating more clearly and briefly what follows."

The original opening paragraph to the "seven" lectures probably started with the words, "The matters we are going to consider, etc.," the first paragraph being afterwards added to connect up the first five lectures.

Barrow indicates that his subject is going to be the consideration of tangents in distinction to the other parts of geometry, which had been already fairly thoroughly treated; he probably alludes to the work on areas and volumes by the method of exhaustions and the method of indivisibles, of which some account has been given in the Introduction; when he treats of areas and volumes himself, he intends to use the work which, by that time, he has done on the properties of tangents. From this we see the reason why the necessity arose for his two theorems on the inverse nature of differentiation and integration.

That Barrow himself knew the importance of what he was about to do is perfectly evident from the next paragraph. He distinctly says that Tangents had been investigated neither thoroughly nor in general; also he claims *distinctly that, to the best of his knowledge and belief, his method is quite original.* He further suggests that it will be found a distinct improvement on anything that had been done before. In other words, he himself claims that he is inventing a new thing, and prepares to write a short textbook on the Infinitesimal Calculus. And he succeeds, no matter whether the style is not one that commended itself to his contemporaries, or whether the work of Descartes had revolutionized mathematical thought; he succeeds in his task. In exactly the same way as the man who put the eye of a needle in its point invented the sewing-machine.

Barrow sets out with being able to draw a tangent to a circle and to a hyperbola whose asymptotes are either given or can be easily found, and the fact that a straight line is everywhere its own tangent. Whenever a construction is not immediately forthcoming from the method of description of the curve in hand, he usually has some means of drawing a hyperbola to touch the curve at any given point; he finds the asymptotes of the hyperbola, and thus draws the tangent to it; this is also a tangent to the curve required. Analytically, for any curve whose equation is $v = f(x)$, he uses as a first approximation the hyperbola $y = (ax+b)/(cx+d)$.

He then gives a construction for the tangent to the general paraboliform, and makes use of these curves as auxiliary curves. As will be found later, he proves that $1+nx$ is an approximation to $(1+x)^n$, leading to the theorem that if $y = x^n$, then $dy/dx = n . y/x$. Thus he founds the whole of his work on exactly the same principles as those on which the calculus always is founded, namely, on the approximation to the binomial theorem; and he does it in a way that *does not call for any discussion of the convergence of the binomial or any other series.*

For the benefit of those who are beginners in mathematical history, it may not be out of place if I here reiterate the warning of the Preface (for Prefaces are so often left unread) that Barrow *knew nothing of the Calculus notation of Leibniz.* Barrow's work is *geometrical*, as far as his published lectures go; the nearest approach to the calculus of to-day is given in the "a and e" method at the end of Lecture X.

Again, with regard to the differentiation of the complicated function, given as a specimen at the end of this volume, I do not say that Barrow ever tackled such a thing. What I do urge, however, is that Barrow could have done so, if he had come across such a function in his own work. My argument, absolutely conclusive I think, is that I have been able to do so, using nothing but Barrow's theorems and methods.

LECTURE VI

Lemmas; determination of curves constructed according to given conditions; mostly hyperbolas.

1. [The opening paragraph, as quoted in the note at the end of the preceding lecture.]

2. Let ABC be a given angle and D a given point; also let the line ODO be such that, if any straight line DN is drawn through D, the length MN, intercepted between the arms of the angle, is equal to the length DO, intercepted between the point D and the line ODO; then the line ODO will be a hyperbola.*

Moreover, if MN is supposed to bear always the same ratio to DO (say a given ratio R : S), the line ODO will be a hyperbola in this case also.

3. Here I note, in passing, that it is easy to solve the problem by which the solutions of the problems of Archimedes and of Vieta were reduced to conic sections by the aid of a previously constructed conchoid.

Thus "to draw through a given point D a straight line, so that the part of the straight line so drawn, intercepted between the arms of a given angle ABC, may be equal to a given straight line T."

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* There is a very short proof given to this theorem, as an alternative. It is hard to see why the comparatively clumsy first proof is retained, unless the alternative proof was added in revise (? by Newton). There is also a reference to easy alternative proofs for §§ 4, 9. These alternative proofs depend on an entirely different property of the curve.

For, if the hyperbola (of the preceding article) is first described, and if with centre D, and a radius equal to the given straight line T, a circle POQ is described, cutting the hyperbola in O, and DO is produced to cut the arms of the angle in M and N; then it follows that $MN = DO = T$.

4. Let ABC be a given angle and D a given point; and let the line OBO be such that, if through D any straight line DN is drawn, the length MN intercepted between the arms of the angle bears always the same ratio (say $X : Y$) to the length MO intercepted between the arm BC and the curve OBO; then OBO will be a hyperbola.

5. If MO is taken on the other side of the straight line BC, the method of proof is the same.

6. Inference.—If a straight line BQ divides the angle ABC, and through the point D are drawn, in any manner, two straight lines MN, XY, cutting the straight line BQ in the points O, P, of which O is the nearer to B; then

$$MN : MO < XY : XP.$$

7. Moreover, if several straight lines BQ, BG . . . divide the angle ABC, and if from the point D the straight lines DN, DY are drawn, cutting BC, BQ, BG, BA in M, O, E, N and X, V, F, Y, DN being the nearer to B; then $NE : MO < YF : VX$.

8. From what has gone before, it is also evident that through B (in one of two directions) a straight line can be

so drawn that the segments intercepted on lines drawn through D between the constructed line and BC shall have to the segments intercepted between BA and BC a ratio that is less than a given ratio.

9. Again, suppose a given angle ABC and a given point D; also let the line OOO be such that, if through D any straight line DO is drawn, cutting the arms of the angle in M, N, then DM always bears to NO a given ratio (X : Y say;) then the line OOO will be a hyperbola.

10. A straight line ID being given in position, and a point D fixed in it, let DNN be a curve such that, if any point G is taken in ID, and a straight line GN is drawn parallel to a straight line IK given in position, and if two straight lines whose lengths are m and b are taken, and if we put DG $= x$, and GN $= y$, there is the constant relation $my + xy = mx^2/b$; then DNN will be a hyperbola.

11. If the equation is $my - xy = mx^2/b$, the same hyperbola is obtained, only G must be taken in DM instead of DO. If, however, the equation is $xy - my = mx^2/b$, then G must be beyond M and the hyperbola conjugate to the former is obtained.

12. If BDF is a given triangle and the line DNN is such that, if any straight line RN is drawn parallel to BD, cutting the lines BF, DF, DNN in the points R, G, N, and DN is joined; and if DN is then always a mean proportional between RN and NG; then the line DNN is a hyperbola.

13. If ID is a straight line given in position; and DNN is a curve such that, if any point G is taken in ID, and the

straight line GN is drawn parallel to IK, a straight line given in position, and if straight lines whose lengths are g, m, r, are taken; and if we put DG $= x$, and GN $= y$, then there is a constant relation $xy + gx - mv = mx^2/r$; then the line DNN will be a hyperbola.

If the equation is $-yx + gx + my = mx^2/b$, then the same hyperbola is obtained, but the points G must then be taken between B and M (B being the point where the curve cuts the straight line ID); and if the points G are assigned to other positions, the signs of the equation vary. But it is not opportune to go into them at present.

14. Two straight lines DB, DA, are given in position, and along the line DB a straight line CX is carried parallel to BA; also, by turning round the point D as a centre, a straight line DY moves so that, if it cuts BA in X, there is always the same ratio between the lines BE and CD (equal to the ratio of some assigned length R to DB, say); then, if DE cuts CX in N, the line DNN is a parabola.

Gregory St Vincent gave this, *but demonstrated with laborious prolixity*, if I remember rightly.

We add the following:—

15. If, other things remaining the same, CX and DY are moved in such a way that now BE and BC are always in the same ratio (BD : R, say); their intersections will give a parabola also.

16. If, with other things remaining the same, the straight line CX is not now carried parallel to BA, but to some other straight line DH, given in position; and if the ratio

of BE to DC is always equal to the ratio of DB to R; then the intersections N will lie on a hyperbola.

17. Moreover, other things remaining the same as in the preceding, if CX now moves in such a way that BE always bears the same ratio to BC (BD : R, say); the intersections in this case will also lie on a hyperbola.

18. Let two straight lines DB, DA be given in position, and a point D fixed in DB; and let the line DNN be such that, if any straight line GN is drawn parallel to BA, and two straight lines whose lengths are g, r are taken, and DG, GN are called x, y; and if $ry - xy = gx$; then the line DNN will be a hyperbola.

If, however, the equation is $xy - ry = gx$, we must take DE $= r$, and BO $= g$ (measured below the line DB); the proof is the same as before.

19. Let two straight lines DB, BA be given in position; and let the straight line FX move parallel to DB, and let DY pass through the fixed point D, so that the ratio of BE to BF is always equal to an assigned ratio, say DB to R; then the intersections of the straight lines DY, GN lie on a straight line.

20. But if, other things remaining the same, some other point O is taken in AB, which we take as the origin of reckoning, so that the ratio BE to OF is always equal to the ratio DB to R; then the intersections will lie on a hyperbola.

21. Moreover, other things remaining the same, let the straight line FX now move not parallel to DB, but to another straight line DH, so that, a fixed point O being taken in

BA, the ratio BE to OF is always equal to an assigned ratio (say DB to *m*); then the intersections will again lie on a hyperbola.

22. Let ADB be a triangle and DYY a line such that, if any straight line PM is drawn parallel to DB, meeting AB in M, PY is always equal to $\surd(PM^2 - DB^2)$; then the line DYY is a hyperbola.

Cor. If YS is the tangent to the hyperbola DYY, then

$$PM^2 : PY^2 = PA : PS.$$

23. If, other things remaining the same, we have now $PY = \surd(PM^2 + DB^2)$; then the line DYY is again a hyperbola.

Cor. If YS is the tangent to the hyperbola, then

$$PM^2 : PY^2 = PA : PS.$$

24. If ADB is a triangle, having the angle ADB a right angle, and the curve CGD is such that, if any straight line FEG is drawn parallel to DB, cutting the sides of the triangle in F, E, and the curve in G, the rectangle contained by EF and EG is equal to the square on DB; then the curve CGD is an ellipse, of which the semi-axes are AD, AC.

Cor. Let GT be a tangent to the ellipse, then

$$EF^2 : EG^2 = AE : AT.$$

25. If DTH is any rectilineal angle, and A is a fixed point in TD, one of its arms; if also the curve VGG is such that, when any straight line EFG is drawn perpendicular to TD, cutting the lines TD, TH, VGG in the points E, F, G, and AF is joined, EG is always equal to AF; then the line VGG will be a hyperbola.

Note. If a straight line FQ is drawn perpendicular to TH, and QR is taken equal to AE (along TD), and GR is joined; then GR will be perpendicular to the hyperbola VGG.

Take this on trust from me, if you will, or work it out for yourself;* I will waste no words over it.

26. Let two straight lines, AC, BD, intersecting in X, be given in position; then if, when any straight line PKL is drawn parallel to BA, cutting AC, BD, in the points P and K, PL is always equal to BK; then the line ALL will be a straight line.

27. Let a straight line AX be given in position and a fixed point D; also let the line DNN be such that, if through D any straight line MN is drawn, cutting AX in M, and the line DNN in N, the rectangle contained by DM and DN is equal to a given square, say the square on Z; then the line DNN will be circular.

Thus you will see that not only a straight line and a hyperbola, but also a straight line and a circle, each in its own way, are reciprocal lines the one of the other.

But here, although we have not yet finished our preliminary theorems, we will pause for a while.

Note

It has already been noted in the Introduction that the proofs which Barrow gives for these theorems, even in the case where he uses an algebraical equation, are more or less of a strictly geometrical character; the terms of his equations are kept in the second degree, and translated into rectangles to finish the proofs. In this connection, note the remark on page 197 to the effect that I cannot imagine Barrow ever using a *geometrical relation*, in which the expressions are of the fourth degree. The asymptotes of the hyperbolæ are in every case found; and this points to his intention of using these curves as auxiliary curves for drawing tangents; cases of this use will be noted as we come across them; but the fact that the number of cases is small suggests that the paraboliforms, which he uses more frequently, were to some extent the outcome of his researches rather than a first intention.

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* "Ad Calculum exige."—I hardly think that Barrow intends "by analysis," but he may.

The great point to notice, however, in this the first of the originally designed seven lectures, is that the idea of Time as the independent variable, *i.e.* the kinematical nature of his hypotheses, is neglected in favour of either a geometrical or an algebraical relation, as the law of the locus. The distinction is also fairly sharply defined.

Barrow, throughout the theorems of this lecture, gives figures for the particular cases that correspond to rectangular, but his proofs apply to oblique axes as well (in that he does not make any use of the right angle). Both the figures and the proofs have been omitted in order to save space; all the more so, as this lecture has hardly any direct bearing on the infinitesimal calculus.

The proofs tend to show that Barrow had not advanced very far in Cartesian analysis; at least he had not reached the point of diagnosing a hyperbola by the fact that the terms of the second degree in its equation have real factors; or perhaps he does not think his readers will be acquainted with this method of obtaining the asympotes.

LECTURE VII

Similar or analogous curves. Exponents or Indices. Arithmetical and Geometrical Progressions. Theorem analogous to the approximation to the Binomial Theorem for a Fractional Index. Asymptotes.

Barrow opens this lecture with the words, "Hitherto we have loitered on the threshold, nor have we done aught but light skirmishing." The theorems which follow, as he states at the end of the preceding lecture, are still of the nature of preliminary lemmas; but one of them especially, as we shall see later, is of extraordinary interest.

For the rest, it is necessary to give some explanation of Barrow's unusual interpretation of certain words and phrases, *i.e.* an interpretation that is different from that common at the present time. A series of quantities in continued proportion form a Geometrical Progression; thus, if we have A, B, C, D in continued proportion, then these are in Geometrical Progression, and A : B = B : C = C : D. Barrow speaks of these as being "four proportionals geometrically," and this accords with the usual idea. But he also speaks of "four proportionals arithmetically" to signify the four quantities A, B, C, D, which are in Arithmetical Progression; that is, A − B = B − C = C − D; and further his proofs, in most cases, only demand that A − B = C − D. If A, B, C, D, E, F, . . . N and *a*, *b*, *c*, *d*, *e*, *f*, . . . *n* are two sets of proportionals, he speaks of corresponding terms of the two sets as being "*of the same order*"; thus, B, *b*; C, *c*; . . . are "*mean proportionals of the first*, *second*, . . . *order* between A and N, *a* and *n* respectively; and this applies whether the quantities are in Arithmetical or Geometrical Progression.

An *index* or *exponent* is also defined thus:—If the number of terms from the first term, A say, to any other term, F say, is N (excluding the first term in the count), then N is the index or exponent of the term F. Later, another meaning is attached to the word *exponent*; thus, if A, B, C are the general ordinates (or the radii vectores) of three curves, so related that B is always a mean of the same order, say the third out of six means, between A and C; so that the indices or exponents of B and C are 3 and 7 respectively; then 3/7 is called the exponent of the curve BBB. There is no difficulty in recognizing which meaning is intended, as Barrow uses n/m for the latter case, instead of N/M. The connection with the ordinary idea of indices will appear in the note to § 16 of this lecture and that to Lect. IX, § 4.

1. Let A, B be two quantities, of which A is the greater; let some third quantity X be taken; then $A+X : B+X < A : B$.

For, since $X : A < X : B$, $X+A : A < X+B : B$; hence, etc.

2. Let three points, L, M, N be taken in a straight line

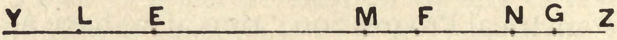


Fig. 61.

YZ; and between the points L, M let any point E be taken, and another point G outside LN (towards Z); let EG be cut in F so that $GE : EF = NL : LM$; then F will fall between M and Z.

$$\text{For } NE : ME > NL : ML\ (= GE : EF) > NE : FE,$$
$$\therefore FE > ME.$$

3. Let BA, DC be parallel straight lines, and also BD, GP; through the point B draw two straight lines BT, BS,

cutting GP in L and K; then I assert that $DS : DT = KG : LG$

For the ratio $KG : LG$

is compounded of $KG : GB$ and $GB : LG$,

that is, of $PK : PS$ and $PT : PL$,

that is, of $DB : DS$ and $DT : DB$,

and hence is equal to the ratio $DT : DS$.

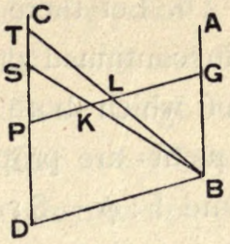
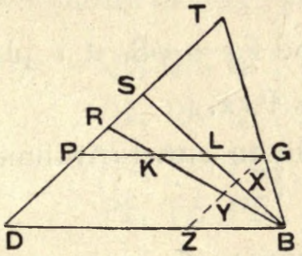
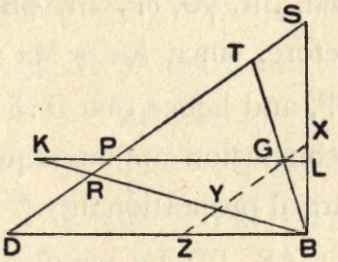


Fig. 62.

4. Let BDT be a triangle, and let any two straight lines BS, BR, drawn through B, meet any straight line GP drawn parallel to the base BD in the points L, K; then I say that

$$LG . TD + KL . RD : KG . TD = RD : SD.$$





[Barrow's proof by drawing parallels through L is rather long and complicated; the following short proof, using a different subsidiary construction, is therefore substituted.

Construction.—Draw $GXYZ$, as in the figures, parallel to TD.

Proof.— Since $LG : BZ = GX : XZ = TS : SD$,

and $KG : BZ = GY : YZ = TR : RD$;

hence $RD : SD = LG . TR : KG . TS$

$= LG . TR + RD . KG : KG . TS + KG . SD$

$= LG . TD + KL . RD : KG . TD$.]

5. But, if the points R, S are not situated on the same side of the point D,* then by a similar argument,

$$LG . TD - KL . RD : KG . TD = RD : SD.$$

---

* This is a mistake: D should be P, and $LG . TD - KL . RD$ should be ambiguous in sign.

6. Let there be four equinumerable series of quantities in continued proportion (such as you see written below), of which both the first antecedents and the last consequents are proportional to one another (*i.e.* $A : \alpha = M : \mu$, and $F : \phi = S : \sigma$); then, the four in any the same column being taken, they will also be proportional to one another (say, for instance, $D : \delta = P : \pi$).

$$\left.\begin{array}{cccccc} A, & B, & C, & D, & E, & F \\ \alpha, & \beta, & \gamma, & \delta, & \epsilon, & \phi \\ M, & N, & O, & P, & R, & S \\ \mu, & \nu, & o, & \pi, & \rho, & \sigma \end{array}\right\}$$

For $A\mu$, $B\nu$, Co, $D\pi$, $E\rho$, $F\sigma$ and αM, βN, γO, δP, ϵR, ϕS } are in continued proportion.

Therefore, since $A\mu = M\alpha$ and $F\sigma = \phi S$, it is plain that $D\pi = \delta P$, and hence that $D : \delta = P : \pi$.

The conclusion applies equally to either Arithmetical or Geometrical proportionality.*

7. Let AB, CD be parallel lines; and let a straight line BD, given in position, cut these. Now let the curves EBE, FBF be so related that, if any straight line PG is drawn parallel to BD, PF is always a mean proportional of the same

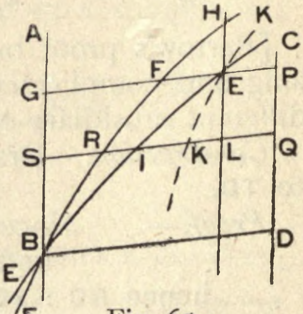


Fig. 65.

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* If the series are "arithmetical proportionals," then

$A+\mu$, $B+\nu$, $C+o$, $D+\pi$, $E+\rho$, $F+\sigma$ / $\alpha+M$, $\beta+N$, $\gamma+O$, $\delta+P$, $\epsilon+R$, $\phi+S$ } are arithmetical proportionals; and the condition for the first antecedents and the last consequents must be $A+\mu = \alpha+M$ and $F+\sigma = \phi+S$; in this case $D+\pi = \delta+P$, or $D-\delta = P-\pi$.

This is not of very great importance, as, in the following theorems, Barrow apparently only considers geometrical proportionals. For arithmetical proportionality, the lines AGB, HEL must be parallel curves, *i.e.* EG must be constant, and then the curves KEK and FBF are parallel curves, *i.e.* RK is constant.

order between PG and PE; then, through any point E of the given line EBE, let HE be drawn parallel to AB and CD; and let another curve KEK be such that, if any straight line QL is drawn also parallel to BD, cutting EBE in I, HE in L, FBF in R, and CD in S, then QK is always a mean proportional of the same order between QL and QI (of that order, I say, of which PF was a mean proportional between PG and PE); then I assert that the lines FBF, KEK are similar; that is, the ordinates such as QR, QK bear a constant ratio to one another, the ratio which PF bears to PE.

This follows from the lemma just proved, as will be clear by considering the argument below.

$$\text{Since}\quad \left.\begin{array}{l} QS,\ QR,\ QI\\ QL,\ QK,\ QI\\ PG,\ PF,\ PE\\ PE,\ PE,\ PE\end{array}\right\} \quad \begin{array}{l}\text{are proportionals}\\ \text{such that}\\ QS:QL = PG:PE,\\ QI:QI = PE:PE;\end{array}$$

$$\text{hence,}\qquad QR:QK = PF:PE$$

NOTE. Instead of the straight lines AB, HE, CD, we can substitute any parallels we please, even curved lines.

8. Again, let AQPB, ASGD be two straight lines meeting in A, and let BD be a straight line given in position; also let EBE, FBF be two curves so related that, if any straight line PG is drawn parallel to BD, PF is always a mean proportional of the same order between PG and PE; then, having joined AE, let another curve KEK be such that, if any straight line QLI is drawn parallel to BD, cutting AE in L, EBI in I, and FBF in R, QK is always a mean proportional

between QL and QI of the same order as PF was between PG and PE; then the line FBF is similar to the line KEK; that is, QR : QK = PF : PE, in every position.

NOTE. For the three straight lines AB, AH, AD we can substitute any three analogous lines.

9. Also, if AGB is a circle whose centre is D; and EBE, FBF are two other curves such that, if any straight line DG is drawn through D, DF is always a mean proportional of the same order between DG and DE; then, through E, let a circle HE, with centre D, be drawn; and let another curve KEK be drawn such that, if any straight line DL is drawn through D to meet the circle HE in L, and EBE in I, DK is always a mean proportional between DL and DI, of the same order as DF was between DG and DE; then the curves FBF, KEK will be similar, *i.e.* DR : DK = DF : DE, in every position. [DL meets FBF in R.]

NOTE. In this case also, instead of the circles, we may substitute any two parallel or two analogous lines.

10. Lastly, let AGBG, EBE be any two lines; and let FBF be another line so related to them that, if any straight line DG be drawn in any manner through a fixed point D, DF is always a mean proportional of the same order between DG and DE; then let HEL be a line analogous to AGB (*i.e.* such that, if through D a straight line DSL is drawn in any manner, DS and DL are always in the same proportion); lastly, let the line KEK be such that, if DL be drawn in any manner, cutting EBE in I, DK is always a mean proportional between DL and DI, of the same order as DF was between

DG and DE; then, in this case also, FBF is analogous to the line KEK.

11. Let A, B, C, D, E, F be a series of quantities in Arithmetical Progression; and, two terms D, F being taken in it, let the number of terms from A to D (excluding A) be N, and the number of terms from A to F (excluding A) be M; then $A \sim D : A \sim F = N : M$.

For, suppose the common difference to be X; then

$$D = A \pm N . X \quad \text{and} \quad F = A \pm M . X;$$

therefore $A \sim D : A \sim F = N . X : M . X = N : M$.

12. Hence, if there are two series of this kind, and in each a pair of terms, corresponding to one another in order, are taken (say D, F in the first, and P, R in the second); then $A \sim D : A \sim F = M \sim P : M \sim R$,

where the series are

A, B, C, D, E, F and M, N, O, P, Q, R.

For each of these ratios is equal to that which the numbers, N, M, as found in the preceding article, bear to one another.

These numbers N, M, in any series of proportionals, we shall usually call the *exponents* or *indices* of the terms to which they apply; and where we use these letters in what follows, we shall always understand them to have this meaning.

13. Let any quantities A, B, C, D, E, F be a series in Arithmetical Progression; and let there be another set, equal in number, in Geometrical Progression, starting

with the same term A; thus, [suppose the two series are]

A, B, C, D, E, F.

A, M, N, O, P, Q.

Also let the second term B of the Arithmetical Progression be not greater than M the second term of the Geometrical Progression; then any term in the Geometrical Progression is greater than the term in the Arithmetical Progression that corresponds to it.*

For, $\quad A+N > 2M > 2B$ or $A+C$, $\therefore N > C$;

hence, $\quad M+N > B+C$ or $A+D$; but $A+O > M+N$;

$\therefore A+O > A+D$, *i.e.* $O > D$;

hence, $\quad M+O > B+D$ or $A+E$; but $A+P > M+O$;

$\therefore A+P > A+E$, *i.e.* $P > E$;

and so on, as far as we please.

14. Hence, if A, B, C, D, E, F are in Arithmetical Progression, and A, M, N, O, P, Q are in Geometrical Progression, and the last term F is not less than the last term Q (the number of terms in the two series being equal); then B is greater than M.

For, if we say that B is not greater than M, then F must be less than Q; which is contrary to the hypothesis.

15. Also, with the same data, the penultimate E is greater than the penultimate P.

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* Algebraically:—If the series are a, $a+d$, $a+2d$, $a+3d$, . . . and a, ar, ar^2, ar^3, . . . , we have, whether $r \gtrless 1$, the fact that $(1-r)(1-r)$, $(1-r)(1-r^2)$, $(1-r)(1-r^3)$, . . . are all positive; hence it follows that

$$a+ar^2 > 2ar, \quad a+ar^3 > ar+ar^2, \quad a+ar^4 > ar+ar^3, \ldots$$

Hence, since ar is not less than $a+d$, it follows that

$$a+ar^2 > 2(a+d) \quad \text{or} \quad ar^2 > a+2d,$$
$$a+ar^3 > (a+d)+(a+2d) \quad \text{or} \quad ar^3 > a+3d,$$

and so on, in exact equivalence with Barrow's proof.

16. *Moreover, with the same data, any term in the Arithmetical series is greater than any term in the Geometrical series; for instance,* C > N.

For E > P, and hence D > O, and so on, ∴ C > N.*

17. Hence it may be proved that:—If there are any four lines HBH, GBG, FBF, EBE, cutting one another at B, and these are so related that, if any straight line DH is drawn in any manner parallel to a straight line BD, given in position, or if through a given point D any straight line DH is drawn, DG is always an arithmetical mean of the same order between DH and DE, and DF is the geometrical mean of the same order; then the lines GBG, FBF will touch one another. For it is evident from the preceding that the line GBG will lie wholly outside the line FBF.

18. Hence also (I mention it briefly in passing), the asymptotes, straight lines applying to many different kinds of hyperbolæ, and curves of hyperbolic form, may be defined.†

Thus, let two straight lines VD, BD be given in position, and let AGB, VEI be two other straight lines; now, any straight line PG being drawn parallel to DB, let Pφ be always an arithmetical mean of the same order between PG and PE, and let PF be the geometrical mean of the same order. Now, since the straight lines EG, Eφ are always in the same ratio, the line φφφ is a straight line; but the line VFF is a hyperbola or some curve of hyperbolic form (the hyperbola of Apollonius indeed, if PF is the simple geometrical mean between PG and PE, but some curve of hyperbolic form of a different kind, if PF is a mean of some other kind); and it is plain from the last theorem that the line $\phi\phi\phi$ is an asymptote to the line VFF, corresponding to the points of the same kind of mean.

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* See note at the end of this lecture; the italics are mine.

† See note at the end of the lecture.

I do not know whether this is of very much use, but indeed it was an incidental corollary for us to have obtained it here.

19. Let three straight lines BA, BC, BQ be drawn through a given fixed point B to a straight line AC, fixed in position; then, in QC produced let some point D be taken as a fixed point. Then it is possible to draw through B a straight line (BR say), on either side of BQ, such that, if any straight line is drawn through D, as DN, the part intercepted between BQ and BR is less than the part intercepted between BA and BC.

20. Let D, E, F be three points in a straight line DZ, and let F be the vertex of a rectilineal angle BFC, of which the arms are cut by a straight line DBC; let a straight line EG be drawn through E; then it is possible to draw through E a straight line EH such that, on any straight line DK drawn through the point D, the intercept between the lines EG, EH is less than the intercept between the lines FC, FB.

21. Let a straight line BO touch a curve BA in B; and let the length BO of the straight line be equal to the arc BA of the curve; then, if any point K is taken in the arc

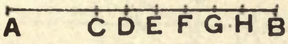
BA and KO is joined, the straight line KO is greater than the arc KA.

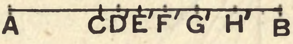
22. Hence, if any two points K, L, on the same side of the point of contact, are taken, one in the curve and the other in the tangent, and KL is joined; then KL + LO > arc KA.

NOTE

From § 16 we have the following geometrical theorem.

Suppose a line AB is divided into two parts at C, and that the part CB is divided at D, E, F, G, H in the figure on the left, and at D′, E′, F′, G′, H′ in the figure on the right, so that AC, AD, AE, AF, AG, AH, AB are in Arithmetical Progression, and AC, AD′, AE′, AF′, AG′, AH′, AB are in Geometrical Progression; then AD > AD′, . . ., AH > AH′.





Expressing this theorem algebraically, we see that, if AC $= a$ and CB $= ax$, and the number of points of section between C and B is $n-1$, and F is the rth arithmetical, and F′ the rth geometrical "mean point" between C and B, then the relation AF > AF′ becomes

$$a + r \,.\, ax/n > a \,.\, [\sqrt[n]{\{(a+ax)/a\}}]^r;$$

i.e. $$1 + (r/n)x > (1+x)^{r/n}, \quad \text{where} \quad r < n.$$

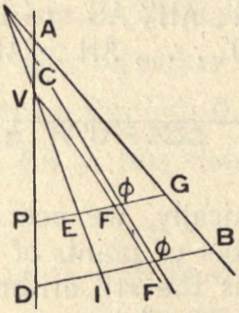
Also, as CB becomes smaller and smaller, the difference FF′ becomes smaller and smaller, since it is clearly less than CB; that is, the ratio FF′/AC can be made less than any assigned number by taking the ratio CB/AC small enough. Hence the algebraical inequality tends to an equality, when x is taken smaller and smaller.

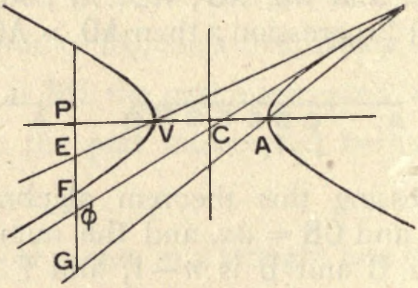
Again, if we put $rx/n = y$, we have $x = ny/r$, and then

$$1 + y > (1 + ny/r)^{r/n} \quad \text{or} \quad 1 + (n/r)y < (1+y)^{n/r},$$

where $n > r$; and again the inequality tends to become an equality if y is taken small enough.

Naturally, a man who uses the notation xx for x^2 does not state such a theorem about fractional indices. But the approximation to the binomial expansion is there just the same, though concealed under a geometrical form. We may as well say that the ancient geometers did not know the expansion for *sin* (A + B), when they used it in the form of Ptolemy's theorem, as say that Barrow was unaware of this. Moreover, if further corroborative evidence is needed, we have it in § 18. Here Barrow states that a line $\phi\phi\phi$ is an asymptote to a curve VFF, the distance between the curve and its asymptote, measured along a line parallel to a fixed direction, being the equivalent of our FF′ in the work above. Now the condition for an asymptote is that this distance should continually decrease and finally become evanescent as we proceed to "infinity." Let us try to reason out the manner in which Barrow came to the conclusion that his line was an asymptote to his curve.





The figure on the left is the one used by Barrow for § 18; as P moves away from V, PE and PG both increase without limit, but it can readily be seen that the ratio of EG to PE steadily decreases. This is all that can be gathered from the figure; and, as far as I can see, it must have been from this that Barrow argued that the distance Fϕ decreased without limit and ultimately became evanescent. In other words, he appreciated the fact that the inequality tended to become an equality when x was taken small enough. Assuming that my suggestion is correct, the very fact that he has recognized this important truth leads him into a trap; for *the line* $\phi\phi\phi$ *is not an asymptote* to the curve VFF, *i.e.* as we understand an asymptote at the present day. Taking the simplest case, as mentioned by Barrow, of the ordinary hyperbola, it is readily seen that the other branch of the curve passes through the common point of the straight lines AGB, VEI, and therefore the line $\phi\phi\phi$ cannot be an asymptote, for it also passes through this common point and touches the curve there.

This is easily seen analytically, taking the figure on the right. For, if the equations of VEI and AGB, referred to VD and a line parallel to DB through the middle point of AV as axes, are $y = n(x+a)$ and $y = m(x-a)$, then the equation to the hyperbola is $y^2 = mn(x^2 - a^2)$; that of the asymptote, with which Barrow confuses the line $\phi\phi\phi$, is $y = \sqrt{(mn)}\,.\,x$; and that of the line $\phi\phi\phi$ is $2y = (m+n)x + (m-n)a$; and the two lines are not the same unless $m = n$, *i.e.* unless VEI and AGB are parallel. The argument is the same, if DB is not taken at right angles to VD, or for different kinds of "means."

The true source of the error is, of course, that it is not true that Fϕ decreases without limit, but that it is Fϕ : PE which decreases without limit, whilst PE increases without limit. This kind of difficulty is exactly on a par with the difficulties arising from considerations of convergence of infinite series. Barrow certainly has in his theorem the equivalent of the binomial approximation as far as it is necessary for differentiation of fractional powers in the ordinary method; it is very likely that he may have found difficulties with other theorems of the kind discussed above; but, as will be seen in the note to Lect. IX, § 4, he is quite independent of considerations of this sort, *i.e.* of infinite series with all their difficulties; for all that he requires is *the bare inequality*, as given in his theorem. By means of this, *at the very least*, he was the *first man* to give a *rigorous demonstration* of a method for differentiating a *fractional power* of the variable.

As an example of the use that a geometer could make of his geometrical facts, it may be pointed out that the theorem of § 17 is equivalent to the analytical theorem :—

The curves whose equations are

$y = [\{n-r\}\,.f(x) + r\,.\,F(x)]/n$ and $y = \sqrt[n]{[\{f(x)\}^{n-r}\,.\,\{F(x)\}^r]}$

touch one another at all the points common to the two curves whose equations are $y = f(x)$ and $y = F(x)$.

LECTURE VIII

Construction of tangents by means of auxiliary curves of which the tangents are known. Differentiation of a sum or a difference. Analytical equivalents.

Truly I seem to myself (and perhaps also to you) to have done what that wise man, the Scoffer,* ridiculed, namely, to have built very large gates to a very small city. For up to the present, we have done nothing else but struggle towards the real thing, just a little nearer. Now let us get to it.

1. We assume the following :—

If two lines OMO, TMT touch one another, the angles between them (OMT) are less than any rectilineal angle; and conversely, if two lines contain angles which are less than any rectilineal angle, they touch one another (or at least, they will be equivalent to lines that touch).

The reason for this statement has already been discussed, unless I am mistaken.

2. Hence, if any third line PMP touch two lines OMO, TMT, the lines OMO, TMT will also touch one another.

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* Socrates, the Athenian philosopher : Zeno called him "Scurra Atticus," the Attic Scoffer.

3. Let a straight line FA touch a curve FX in F; and let FE be a straight line given in position; also let EY, EZ be two curves such that, if any straight line IL is drawn parallel to EF, cutting FA in G and the curves FX, EY, EZ in I, K, L respectively, the intercept KL is always equal to the intercept IG; then the curves EY, EZ touch one another.

4. Again, let a straight line AF touch a curve AX, and let EY, EZ be two curves such that, if through a fixed point D any straight line DL is drawn, cutting the given lines as in the preceding theorem, KL is always equal to IG; then the curves EY, EZ will touch one another.

The two foregoing conclusions are also true, and can be shown to be true by a like reasoning, if it is given that IG, KL always bear to one another any the same ratio.

5. Let TEI be a straight line, and let two curves YFN, ZGO be so related that, if any straight line EFG is drawn parallel to AB, a straight line given in position, the intercepts EG, EF always bear to one another the same ratio; also let the straight line TG touch ZGO, one of the curves in G and meet IE in T; then TF, being joined, will touch the curve YFN.

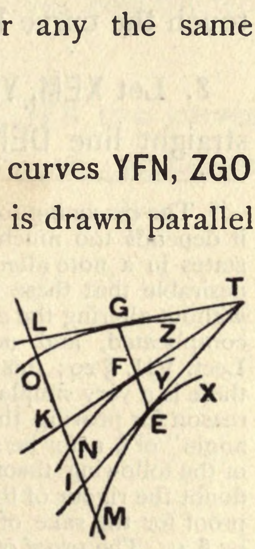

Fig. 80.

For, let a straight line IL, parallel to AB, be drawn, cutting the given lines as shown in the figure. Then

$$IL : IN > IO : IN > EG : EF > IL : IK, \text{ and } \therefore IN < IK;$$

hence the line TF falls altogether without the curve YFN.[*]

Otherwise. It can be shown that IL : IK = OL : NK; hence, by § 4 above, since GL, GO touch, FN, FK also touch.

6. Moreover, if three curves XEM, YFN, ZGO are so related that, if any straight line EFG is drawn parallel to a line given in position, EG and EF are always in the same ratio; also let the tangents ET, GT to the curves XEM, ZGO meet in T; then TF, being joined, will touch the curve YFN.[†]

7. Let D be a given point, and let XEM, YFN be two curves so related that, if through D any straight line DEF is drawn, the straight lines DE, DF always bear to one another the same ratio; and let the straight line FS touch YFN, one of the curves, and let ER be parallel to FS; then ER will touch the curve XEM.[‡]

8. Let XEM, YFN, ZGO be three curves such that, if any straight line DEFG is drawn through a given point D, the

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[*] The reasoning for these theorems given by Barrow is not conclusive; it depends too much on the accident of the figure drawn. Although he states in a note after § 6 that he always chooses the simplest cases, it is desirable that these simple cases should be capable of being generalized without altering the argument. In addition, his proof of § 4 is long and complicated, and necessitates as a preliminary lemma the theorem of Lect. VII, § 20; this is also proved in a far from simple manner, although there is a very simple proof of it. Still Barrow must have had some good reason for proving these two theorems by the method of "the vanishing angle" of § 1, for he states that "these theorems are set forth, so that none of the following theorems may be hampered with doubts." He seems to doubt the rigour of the method used in § 5, of which I have given the full proof for the sake of exemplification; together with the alternative proof by § 4. The proof of the latter follows easily thus:—Since FA lies wholly on one side of the curve FX, EZ lies wholly on one side of EY.

[†] If $y = f(x)$, $y = F(x)$, $y = \phi(x)$ are three curves such that there is a constant relation $A.f + B.F + C.\phi = 0$, where $A + B + C = 0$, the tangents at points having equal abscissæ are concurrent.

[‡] Homothetic curves have parallel tangents; this theorem and the next are the polar equivalents of those of §§ 5, 6.

intercepts EG, EF always bear the same ratio to one another (say as R is to S); and let the straight lines ET, GT touch two of the curves (say XEM, YFN) in E and G; it is required to draw the tangent at F to the curve YFN.

Imagine a curve TFV such that, if a straight line is drawn in any manner through D, cutting the straight lines TE, TG in the points I, L and the curve in K, the intercepts IL, IK bear to one another the same given ratio, R to S. Then IK > IN, and therefore the curve TFK touches the curve YKN. But, by Lect. VI, § 4, the curve TFK is a hyperbola;* let FS be the tangent to it. Then SF will touch the curve YFN also.

Since mention is here made for the first time of a tangent to a hyperbola, we will determine the tangent to this curve, together with the tangents of all other curves constructed by a similar method, or of reciprocal lines.

9. Let VD be a straight line, and XEM, YFN two curves so related that, if any straight line EDF is drawn parallel to a line given in position, the rectangle contained by DE, DF is always equal to any the same area; also the straight line

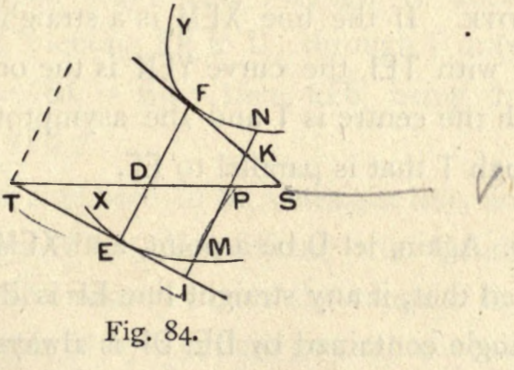

Fig. 84.

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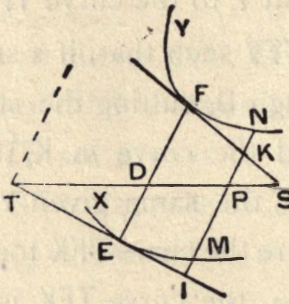
* Note the use of the auxiliary hyperbola.

ET touches the curve XEM at E, and cuts VD in T; then, if DS is taken equal to DT and FS is joined, FS will touch the curve YFN at F.*



Let any straight line IN be drawn parallel to EF, cutting the given lines as shown; then

$$TP : PM > TP : PI > TD : DE; \text{ also } SP : PK = DS : DF;$$

$$TP . SP : PM . PK > TD . SD : DE . DF > TD . SD : PM . PN.$$

But, since D is the middle point of TS, $\therefore TD . SD > TP . SP$ hence all the more, $TD . SD : PM . PK > TD . SD : PM . PN$,

$$\therefore PM . PK < PM . PN \quad \text{or} \quad PK < PN.$$

Therefore the whole line FS lies outside the curve YFN.

NOTE. If the line XEM is a straight line, and so coincident with TEI, the curve YFN is the ordinary hyperbola, of which the centre is T and the asymptotes are TS and a line through T that is parallel to EF.

10. Again, let D be a point, and XEM, YFN two curves so related that, if any straight line EF is drawn through D, the rectangle contained by DE, DF is always equal to a certain

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* The analytical equivalent of this is:—
If y is a function of x, and $z = A/y$, then $(1/z) . dz/dx = -(1/y) . dy/dx$. Also the special case gives $d(1/x)/dx = -1/x^2$. It is thus that Barrow starts his real work on the differential calculus.

square (say the square on Z); and let a straight line ER touch one curve XEM; then the tangent to the other is found thus:—

Draw DP perpendicular to ER and, having made DP : Z = Z : DB, bisect DB at C; join CF and draw FS at right angles to CF; then FS will touch the curve YFN.

11. Let XEM and YFN be two curves such that, if any straight line FE is drawn parallel to a straight line given in position, it is always equal to a given length; also let a straight line FS touch the curve YFN; then RE, being drawn parallel to FS, will touch the curve XEM.*

12. Let XEM be any curve, which a straight line ER touches at E; also let YFN be another curve so related to the former that, if a straight line DEF is drawn in any manner through a given point D, the intercept EF is always equal to some given length Z; then the tangent to this curve is drawn thus:—

Take DH = Z (along DEF), and through H draw AH perpendicular to DH, meeting ER in B; through F draw FG parallel to AB; take GL = GB; then LFS, being drawn, will touch the curve YFN.†

NOTE. If XEM is supposed to be a straight line, and so coincide with ER, then YFN is the ordinary true Conchoid, or the Conchoid of Nicomedes; hence the tangent to this curve has been determined by a certain general reasoning.

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* The analytical equivalent is:—If y is a function of x, and $z = y + c$, where c is a constant, then $dz/dx = dy/dx$.

† For the proof of this theorem, Barrow again uses an auxiliary curve, namely the hyperbola determined in Lect. VI, § 9.

13. Let VA be a straight line, and BEI any curve; and let DFG be another line such that, if any straight line PFE is drawn parallel to a line given in position, the square on PE is equal to the square on PF with the square on a given straight line Z; also let the straight line TE touch the curve DEI; let $PE^2 : PF^2 = PT : PS$; then FS, being joined, will touch the curve DFG.*

[This is proved by the use of Lect. VI, § **22**, and corollary.]

14. Let other things be supposed the same, but now let the square on PE together with the square on Z be equal to the square on PF; also let $PE^2 : PF^2 = PT : PS$; then FS will touch the curve GFG.†

[For this, Barrow uses Lect. VI, § 23, and its corollary.]

15. Let AFB, CGD be two curves having a common axis AD, so related to one another that, if any straight line EEG is drawn perpendicular to AD, cutting the lines drawn as shown, the sum of the squares on EF and EG is equal to the square on a given straight line Z; also let the straight line FR touch AFB, one of the curves; and let $EF^2 : EG^2 = ER : ET$; then GT, being joined, will also touch the curve CGD. ‡

[For this, Barrow uses Lect. VI, § 24, and its corollary.]

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* The analytical equivalent is :—If y is any function of x, and $z^2 = y^2 - a^2$, where a is some constant, then $z . dz/dx = y . dy/dx$; or in a different form, if $z = \sqrt{(y^2 - a^2)}$, then $dz/dx = y . (dy/dx)/(y^2 - a^2)$. The particular case, when $y = x$, is the equivalent of Lect. VI, § 22.

† A similar result for the case of $\sqrt{(y^2 + a^2)}$ or $\sqrt{(x^2 + a^2)}$.

‡ The case of $z = \sqrt{(a^2 - y^2)}$ or $\sqrt{(a^2 - x^2)}$. Since T, R are to be taken on opposite sides of FG.

16. Let AFB be any curve, of which AD is the axis and DB is applied to AD; also let VGC be another curve so related that, if any straight line ZF is drawn through some fixed point Z in the axis AD, and through F a straight line EFG is drawn parallel to DBC, EG is equal to ZF; also let FQ be at right angles to the curve AFB; along AD, in the direction ZE, take QR = ZE; then RG, being joined, will be perpendicular to the curve VGC.

[For this, Barrow makes use of the hyperbola of Lect. VI, § 25, as the auxiliary curve; he did not give a proof of the theorem of that article, but left it "to the reader."]

17. Let DP be a straight line, and DRS, DYX two curves so related that, if any straight line REY is drawn parallel to a straight line DB, given in position, cutting DP in E and the curves DRS, DYX in R, Y, the ratio RY : DY is always equal to the ratio DY : EY; also let the straight line RF touch the curve DRS at R. It is required to draw the tangent to the curve DYX at Y.

Suppose the line DYO is such that, if any straight line GO is drawn parallel to DB, cutting the lines FR, FP, DYO in the points G, P, O, and DO is joined, GO : DO = DO : PO; then the curve DYO touches the curve DYX at Y.

But, in Lect. VI, § 12, it has already been shown that the curve DYO is a hyperbola; let YS touch the hyperbola; then YS also touches the curve DYX.

NOTE. If the curve DRS is a circle, and the angle GDB is a right angle, the curve DYX is the ordinary Cissoid; and thus the tangent to it (together with many other curves similarly produced) is determined.

18. Let DB, VK be two lines given in position, and let the curve DYX be such that, if from the point D any straight line DYH is drawn, cutting the straight line BK in H and the curve DYX in Y, the chord DY is always equal to the straight line BH; it is required to draw the straight line touching the curve DYX in Y.

With centre D and radius DB, describe the circle BRS; let YER, drawn parallel to KB, meet the circle in R; join DR. Then RY : YD = YD : DE; hence, the straight line touching the curve DYX can be found by the preceding proposition.*

19. Let DB, BK be two straight lines given in position; also let BXX be a curve such that, if from a point D any straight line is drawn, cutting BK in H and the curve BXX in X, HX is always equal to BH; it is required to draw the tangent to the curve BXX at X.

Suppose that DYY is a curve such that DY is always equal to BH (such as we considered in the previous proposition), and let YT touch this curve in Y, and cut BK in R; then let the hyperbola NXN be described, with asymptotes RB, RT, to pass through X; then the hyperbola NXN touches the curve BXX at X. Thus, if the tangent to the hyperbola, XS is drawn; XS will also touch the curve BXX.

However, we seem to have trifled with this succession of theorems quite long enough for one time; we will leave off for a while.

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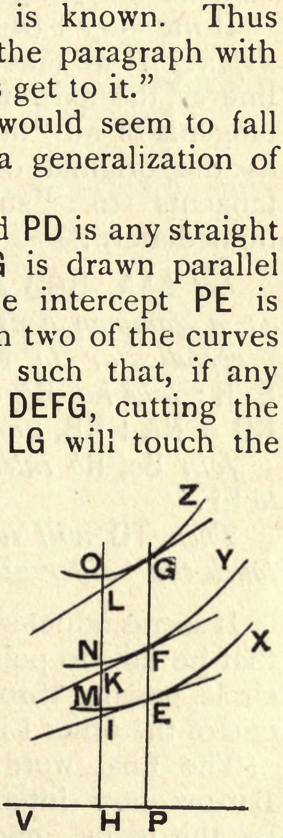
* In § 17, it is not essential that the curve RS should pass through D; hence this statement is justifiable.

NOTE

In the footnote to § 9, I state that it is in this theorem that Barrow starts his real work on the infinitesimal calculus. Certainly he has given theorems on tangents before this point, which have had analytical equivalents; but these have been special cases. Here for the first time he gives theorems that are equivalent to the differentiation of *general functions*, not only of the variable simply, but of any other function that is itself a function of the variable. Thus, the theorem of Lect. VI, § 22 is indeed equivalent to the differentiation of $\sqrt{(x^2 - a^2)}$ with regard to x; but it is in the theorem of Lect. VIII, § 13 that he gives the equivalent to the differentiation of $\sqrt{(y^2 - a^2)}$ with regard to x, where y is *any* function of x whose gradient is known. Thus Barrow substantiates the last words of the paragraph with which he opens the lecture: "Now let us get to it."

He however omits a theorem, which would seem to fall naturally into place in this lecture, as a generalization of the theorem of § 11.

If XEM, YFN, ZGO are three curves and PD is any straight line such that, if any straight line PEFG is drawn parallel to a straight line given in position, the intercept PE is always equal to FG; also let EI, FK touch two of the curves XEM, YFN; draw the straight line GL such that, if any straight line HO is drawn parallel to DEFG, cutting the given lines as shown, KL = HI; then LG will touch the curve LGO.



For, if the two curves XEM, YFN are both convex to the line VP,

since HM = NO, and HI = KL,

∴ $KO > NO > HM > HI > KL$;

hence the curve lies altogether above the line GL.

If both curves are concave to VP, the argument is similar, but now ZGO falls altogether below the line GL.

If one of the curves is concave and the other convex to VP, say XEM, yFn, draw the curve YFN so that the intercept KN is always equal to the intercept nK; then the two curves YFN, yFn touch and have a common tangent KF. Let now the third curve be zGo; then, since $LO = IM + KN$, and $Lo = IM - KN$, therefore oO is always equal to $2KN$; hence, by § 3 above, the curves zGo, ZGO also touch, and LG is the common tangent. Therefore the construction holds in this case also.

I believe the omission of the theorem was intentional; and I argue from it that Barrow himself was not completely satisfied with the theorems of §§ 3, 4, thus corroborating my footnote. This theorem is of course equivalent to the differential of a sum. Barrow may have thought it evident, or he may have considered it to be an immediate consequence of his differential triangle; but I prefer to think that he considered it as a corollary of the theorem of § 5. For this may be given analytically as:—

If $nw = ry + (n - r)z$, then $n \, . \, dw/dx = r \, . \, dy/dx + (n - r) \, . \, dz/dx$. If we take one curve a straight line, and this straight line as the axis, we have $d(ky)/dx = k \, . \, dy/dx$, or the subtangents of all "multiple" curves have the same subtangent as the original curve. Hence the constructions for the tangents to "sum" and "difference" curves follow immediately:—

Let AAA, BBB *be any two curves, of which* EF *is taken as a common axis; let* NAB *be any straight line applied perpendicular to* EF; *let the tangents* AS, BR, *cut* EF *in* S, R; *take* Aa, B*b* *equal to* NA, NB *respectively, and also let* $NC = NA + NB$, *and* $ND = NA - NB$.

Join Sa, R*b* *intersecting in* T, *and draw* TV *perpendicular to* EF.

Then TC *will touch the "sum" curve* CCC, *and* VD *will touch the "difference" curve* DDD.

It seems rather strange, considering Barrow's usual custom, that he fails to point out that, in § 12, if the curve XEM is a circle passing through D, the curve YFN is the Cardioid or one of the other Limaçons.

The final words of the lecture seem to indicate that Barrow now intends to proceed to what he considers to be the really important part of his work; and, in truth, this is what the next lecture will be found to be.

LECTURE IX

Tangents to curves formed by arithmetical and geometrical means. Paraboliforms. Curves of hyperbolic and elliptic form. Differentiation of a fractional power, products and quotients.

We will now proceed along the path upon which we started.

1. Let the straight lines AB, VD be parallel to one another; and let DB cut them in a given position; also let the lines EBE, FBF pass through B, being so related that, if any straight line PG is drawn parallel to DB, PF is always an arithmetical mean of the same order between PG and PE;

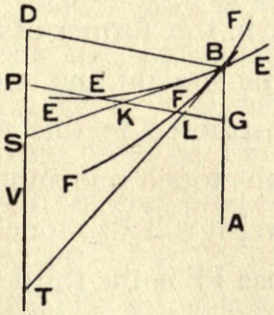
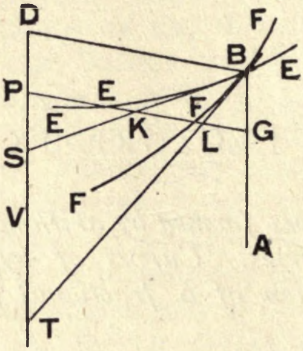


Fig. 94.

and let the straight line BS touch the curve. Required to draw the tangent at B to the curve FBF.

Let the numbers N, M (as explained in Lect. VII, § 12) be the exponents of the proportionals PF, PE; take DT, such that N : M = DS : DT, and join TB; then TB touches the line FBF.



For, in whatever position the line PG is drawn, cutting the given lines as shown in the figure, we have

$$FG : EG = N : M = DS : DT = LG : KG.$$

Hence, since by hypothesis $KG < EG$, ∴ $LG < KG$; and thus it has been shown that the straight line TB falls wholly without the curve FBF.*

2. All other things remaining the same, let now PF be a geometrical mean between PG and PE (namely, the mean of the same order as in the former case of the arithmetical mean); then the same straight line touches the curve FBF.

For the lines constructed in this way from arithmetical and geometrical means touch one another; hence, since BT touches the one curve, it will also touch the other.†

Example.—Suppose PF is the third of six means between PG and PE, then M = 7, and N = 3; and DS : DT = 3 : 7.

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* Note that in this case, FG : EG = LG : KG; and thus this is a particular case of the curves in Lect. VIII, § 5; the analytical equivalent is $d(a+by) = b \,.\, dy/dx$.

† Analytical equivalent:—If y is *any* function of x, and $z^n = a^{n-r} \,.\, y^r$, then $dz/dx = (r/n) \,.\, dy/dx$, when $z = y = a$.

3. Again, the preceding hypothesis being made in all other respects, let *any* point F be taken in the curve FBF;

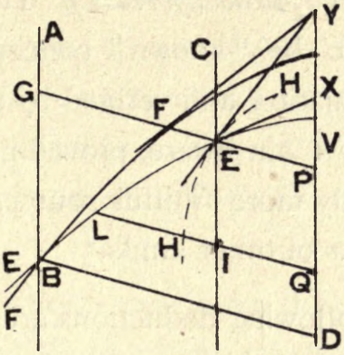

Fig. 95.

then a straight line touching the curve may be drawn by a similar method. Thus, let the straight line PG be drawn through F parallel to DB, cutting the curve EBE in E, and let EX touch the curve EBE at E; take PY, such that N : M = PX : PY, and join FY.

Then the straight line FY touches the curve FBF.*

For, if through E the straight line CEI is drawn parallel to AB or VD, and it is supposed that a curve HEH passing through E is such that, if any straight line QI is drawn parallel to DB, cutting the curves EBE, HEH in L, H, and the straight lines CE, VP in I, Q, QH is a mean between QI and QL of the same order as PF was between PG and PE; then it follows from the preceding proposition that, if YE is joined, it will touch the curve HEH.

But the curves HEH, FBF are analogous curves (Lect. VII, § 7); therefore YF touches the curve FBF (Lect. VIII, § 5).

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* This is a generalization of the last theorem; the equivalent is that, *in general*, if $z^n = a^{n-r} . y^r$, then $(1/z) . dz/dx = (n/r) . (1/y) . dy/dx$. The analogy of the curves occurs in the case of the arithmetical means, for then IH : HL = GF : FE.

4. Note that, if the line EBE is supposed to be straight, then the line FBF is one of the parabolas or curves of the form of a parabola ("*paraboliforms*"). Therefore, that which is usually held to be "known" concerning these curves (deduced by calculation and verified by a sort of induction, but not, as far as I am aware, proved geometrically) flows from an immensely more fruitful source, one which covers innumerable curves of other kinds.*

5. Hence the following deductions are evident:—

If TD is a straight line and EEE, FFF are two curves so related that, when straight lines PEF are drawn parallel to BD, a straight line given in position, the ordinates PE vary as the squares on the ordinates PF; and if ES, FT, straight lines drawn from the ends of the same common ordinate, touch these curves; then TP = 2SP. But, if the ordinate PE varies as the cube of PF, then TP = 3SP; if PE varies as the fourth power of PF, then TP = 4SP; and so on in the same manner to infinity.†

6. Again, let AGB be a circle, with centre D and radius DB, and let EBE, FBF be two lines passing through B, so related to one another that, when any straight line DG is drawn through D, DF is always an arithmetical mean of the same order between DG and DE; also let the straight

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* See note at the end of this lecture; where it is shown that this theorem is equivalent to a *rigorous* demonstration of the method for differentiating a fractional power of the variable.

† This is a special case of the preceding theorem; for PF is the simple geometrical mean between PE and a definite length PG; or the second of two, the third of three, etc., geometrical means between PE and PG; thus $PF^2 = PE . PG$, $PF^3 = PE, PG^2$, etc. This enables Barrow to differentiate any power or root of $f(x)$, when he can differentiate $f(x)$ itself.

line BO touch the curve EBE at B; required to draw the tangent at B to the curve FBF.

This (demonstrated generally, to a certain extent,* in Lect. VIII, § 8) will here be specially shown to follow more clearly and completely from the method above. Thus:—

Let DQ be perpendicular to DB, cutting BO in S; and let N : M = DS : DT; join BT. Then BT touches the curve FBF.†

7. Hence, other things remaining the same as before, if the straight line DF is always taken as a geometrical mean (of the same order as before) between DG and DE, the same straight line BT will touch the curve FBF also.

For the lines formed from arithmetical means and from geometrical means of the same order touch one another, and have a common tangent.

8. Further, other things remaining the same as in the preceding proposition, let *any* point P be taken in the curve FBF; then the straight line that touches the curve at this point can be determined by a similar plan.

Let the straight line DF be drawn, cutting the curve EBE in E; also draw DQ perpendicular to DG cutting EO the tangent to EBE in X; make DX : DY = N : M; join EY, and draw FZ parallel to EY. Then FZ touches the curve FBF.

Hence, not only the tangents to innumerable spirals, but also those to a boundless number of others of different kinds, can be determined quite readily.

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* The actual construction for the asymptotes or tangent to the auxiliary hyperbola is not given.

† Barrow proves his construction by the use of an auxiliary hyperbola using Lect. VI, § 4, and VIII, § 9.

9. Hence, it is clear that, if two lines EEE, FFF are so related that, when any straight line DEF is drawn from a fixed point D, DE varies as the square on DF; and if ES, FT are the tangents to the curves at the ends E, F, meeting the line perpendicular to DEF in S, T; then DT = 2DS. But, if DE varies as the cube of DF, DT = 3DS; and so on.*

10. Let VD, TB be two straight lines meeting in T, and let a straight line BD, given in position, fall across them; also let the lines EBE, FBF pass through B and be such that, if any straight line PG is drawn parallel to BD, PF is always an arithmetical mean of the same order between PG and PE; also let BR touch the curve EBE. Required to draw the tangent at B to the curve FBF.

see intr. p. 29.

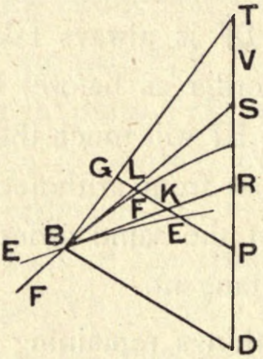

Fig. 100.

Taking numbers N, M to represent the exponents of PF, PE, make N . TD + (M − N) . RD : M . TD = RD : SD, and join BS; then BS touches the curve FBF.†

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[*] As the theorems of §§ 6, 7, 8, 9 are only the polar equivalents of §§ 1, 2, 3, 5, the figures and proofs are not given; their inclusion by Barrow suggests that he was aware of the fact that, with the usual modern notation, $\tan \phi = r . d\theta/dr$.

[†] The form of the equation suggests logarithmic differentiation: see note at end of this lecture.

For, if any straight line PG is drawn, cutting the given lines as in the figure, we have $EG : FG = M : N$;

therefore $\qquad FG . TD : EG . TD = N . TD : M . TD$;

also $\qquad EF . RD : EG . TD = (M - N) . RD : M . TD$.

Hence, adding the antecedents, we have

$$\begin{aligned} FG . TD + EF . RD : EG . TD &= N . TD + (M - N) . RD : M . TD \\ &= RD : SD. \end{aligned}$$

Now, $\qquad LG . TD + EF . RD : EG . TD = RD : SD$; VII, § 4,

therefore

$$FG . TD + EF . RD : EG . TD = LG . TD + KL . RD : KG . TD$$

Hence, since $EG > KG$,

$$\therefore FG . TD + EF . RD > LG . TD + KL . RD^{*}$$

$\therefore$ ratio compounded of FG/EF and $TD/RD >$ than that compounded of LG/KL and TD/RD

or, removing the common ratio RD/TD, $\therefore EG/EF > LG/KL$; hence, by *componendo* $EG/EF > KG/KL > EG/EL$ (by Lect. VII, § 1); therefore $EF < EL$, or the point L is situated on the far side of the curve FBF; and thus the problem is solved.

11. Moreover, all other things remaining the same, if PF is supposed to be a geometrical mean of the same order (plainly as in the cases just preceding) the same straight line BS will touch the curve FBF.

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* This is either a very bad slip on Barrow's part, or else he is making the unjustifiable assumption that near B the ratio of LK to FE is one of equality. In either case the proof cannot be accepted. The demonstration can, however, be completed rigorously as follows from the line

$$FG . TD + EF . RD : EG . TD = LG . TD + KL . RD : KG . TD.$$

Hence $\qquad EG/EF : KG/KL = FG/EF + RD/TD : LG/KL + RD/TD$

$$= EF/EF - RD/TD : KL/KL - RD/TD \text{ (\textit{dividendo})};$$

therefore $EG/EF = KG/KL$, or $EG/FG = KG/GL$; hence, since $EG > KG$, it follows that $FG > LG$, *i.e.* L falls without the curve.

Example.—If PF is a third of six means, or $M = 7$, $N = 3$; then

$$3TD + 4RD : 7TD = RD : SD, \quad \text{i.e.} \quad SD = 7TD . RD/(3TD + 4RD).$$

12. It is evident that, if *any* point F whatever is taken on the line FBF, the tangent at F can be drawn in a similar manner. Thus, through F draw the straight line PG parallel to DB, cutting the curve EBE at E, and through E let ER be drawn touching the curve EBE at E; then make

$$N . TP + (M - N) . RP : M . TP = RP : SP,$$

and join SF. Then SF will touch the curve FBF.

13. Note that, if EBE is a straight line (*i.e.* coinciding with the straight line BR), the line FBF is one of an infinite number of hyperbolas or curves of hyperbolic form; and we have therefore included in the one theorem a method of drawing tangents to these, together with innumerable others of different kinds.

14. If, however, the points T, R do not fall on the same side of D (or P), the tangent BS to the curve EBF is drawn by making $N . RD - (M - N) . TD : M . TD = RD : SD$.

15. Hence also the tangents to not only all elliptic curves (in the case when EBE is supposed to be a straight line coinciding with BR), but to an innumerable number of other curves of different kinds, can be determined by the one method.

Example.—If PF is the fourth of four means, *i.e.* $M = 5$, and $N = 4$; then $SD = 5TD . RD/(4RD - TD)$.

NOTE. If it happens that $N . RD = (M - N) . TD$, then DS is infinite; or BS is parallel to VD. Other points may be noticed, but I leave them.

16. Amongst innumerable other curves, the Cissoid and the whole class of cissoidal curves may be grouped together by this method. For, let DSB be a semi-right angle; and let SGB, SEE be two curves so related that, if any straight line GE is drawn parallel to BD, cutting the given lines BS, DS in F, P, PG, PF, PE are in continued proportion; also let the straight line GT touch the curve SGB at G; then the line touching the curve SEE is found by making

$$2TP - SP : TP = SP : RP;$$

and, in every case, if RE is joined, RE touches SEE.

The proof is easy from what has gone before.

Now, if the curve SGB is a circle, and the angle of application, SPG, is a right angle, then the curve SEE is the ordinary Cissoid or the Cissoid of Diocles; otherwise it will be a cissoidal curve of some other kind. But I only mention this in passing, and will not now detain you longer over it.

NOTE

This lecture is remarkable for the important note of § 4. In it, Barrow calls his readers' attention to the fact that he has given a method for drawing tangents to any of the parabolas or paraboliforms; and apparently he refers in more or less depreciative words to the work of Wallis, whilst claiming that his own work is a geometrical demonstration, and therefore rigorous. If we take a line parallel to PG, and DV, as the coordinate axes, and suppose them rectangular or oblique, then $PF^{M} = PG^{M-N} . PE^{N}$ gives $x^{M} = a^{M-N} . y^{N}$, or $y = k . x^{M/N}$, as the general equation to the curve FBF.

Also, $dy/dx = \mathrm{PT}/\mathrm{PF} = (\mathrm{PT}/\mathrm{PS}).(\mathrm{PS}/\mathrm{PF}) = (\mathrm{M}/\mathrm{N}).y/x$; or, if the axes are interchanged, the equation to the curve is $y = c.x^{\mathrm{N}/\mathrm{M}}$, and then $dy/dx = \mathrm{PF}/\mathrm{PT} = (\mathrm{N}/\mathrm{M}).y/x$. Note particularly that the form suggests logarithmic differentiation.

The theorem of § 6 is a particular case of this, in which $\mathrm{N} = 1$, *i.e.* PF is the first of any number of means between PG and PE, and the equations of the curves are $y = k.x^2$, $k.x^3$, $k.x^4$, etc. (the "parabolas" as distinguished from the "paraboliforms").

It seems strange, unless perhaps it is to be ascribed to Barrow's dislike for even positive integral indices, that he does not make a second note to the effect that if the curve EBE is a hyperbola whose asymptotes are VD and a line parallel to PG, then the curves FBF are the hyperboliforms. For, from this particular case, in a manner similar to the foregoing, it follows that if $y = c.x^{-r}$, where r is any positive rational, either greater or less than unity, then $dy/dx = -r(y/x)$. But Barrow probably intends the reciprocal theorem of Lect. VIII, § 9, to be used thus:—If $y = c.x^{-r}$, let $z = 1/y = k.x^r$; then from Lect. VIII, § 9, we have $(1/z).dz/dx = -(1/y).dy/dx$; also from the above, $dz/dx = r.z/x$; hence $dy/dx = (-r).y/x$. I suggest that Barrow found out these constructions by analysis, using letters such as a and e instead of dy and dx, and that the form of the results suggests very strongly that he first *expressed his equation logarithmically*.

Anyway, Barrow was the *first* to give a *rigorous* demonstration of the form of the differential coefficient of x^r, where r is *any rational whatever*. As far as I am aware, it is the only proof that has ever been given, that does not involve the consideration of convergence of infinite series, or of limiting values, in some form or other. Moreover, he gives it in a form, which yields, as a converse theorem, the solution of the differential equation $dy/dx = r.dz/dx$, although, of course, this is not noted by Barrow, simply because he had not the notation.

Again, considering § 8, which is only § 3 with the constant distance between the parallels, PG, replaced by the constant radius, DG, we see that, if DB is the initial line, and the angle BDG is θ, and the angles between the vector DG and the tangents at E and F are ϕ and χ, $DF = R$, $DE = r$, and $DG = a$, then $\tan\phi : \tan\chi = N : M$, and $R^M = a^{M-N} . r^N$; hence $(d\theta/dr)/(d\theta/dR) = dR/dr = (N/M).(R/r)$ and $\tan\phi : \tan\chi = r\, d\theta/dr : R . d\theta/dR$; and I suggest that it was thus that Barrow obtained the construction for this theorem. I go further. Although it is a consequence of a consideration of the whole work, the present place is the most convenient one for me to state my firm conviction that Barrow's drawing of tangents was a *result* of his knowledge of the fundamental principles of a calculus of infinitesimals in an *algebraic form*, which may have been so cumbrous that it was only intelligible to himself when expressed in geometrical form. I fail to see how else he could possibly have arrived at some of his constructions, or elaborated so many of them in the comparatively short time that he had to spend upon them; unless indeed he was a far greater genius than even I am trying to make him out to be. If he had stumbled on the idea in his young days, as might be possible, one could better understand these theorems as being gradually evolved; but we have his own words against this: "The lectures were elicited by my office." Thus I suggest that whilst his geometrical theorems perhaps took definite shape whilst he was Professor of Geometry at Gresham, his knowledge of the elements of the calculus dated from before this time.

Last, but by no means least, the theorems of §§ 10, 11, 12 are modifications of §§ 1, 2, 3, in which a pair of inclined lines are substituted for the pair of parallels. Referring to Fig. 100 on p. 106, take the angle BDT a right angle, and DT as the axis of x, then the relation given is a relation between *subtangents solely*. Further, instead of BT we can take a fixed curve touching BT at B; and we have:—

If $PF^M = PG^{M-N} . PE^N$, then $N/RD + (M-N)/TD = M/SD$.

Also, if we take $Z^{M-1} . PH = PF^M$, we have by § 5, if WD is the subtangent to the locus of H, $1/WD = M/SD$.

This affords a complete rule for *products*, and combining the result with the reciprocal theorem of Lect. VIII, 9, for *quotients* also.

Thus, putting $N = 1$, and $M = 2$, we have for the general theorem of § 12 the remarkably simple results :—

If GGG, EEE *are two curves, and* PEG *is a straight line applied perpendicular to an axis* PRT, *and* GT, ER *are the tangents to* GGG *and* EEE, *then*

(*i*) *If* HHH *is another curve, so related to the other two that* $Z.PH = PE.PG$; *then, if* HW *is the tangent to* HHH, *meeting the axis in* W, $1/PW = 1/PR + 1/PT$; *i.e.* PW *is a fourth proportional to* $PR + PT$, PR, *and* PT.

(*ii*) *If* KKK *is another curve so related to* GGG *and* EEE *that* $PK : Z = PE : PG$, *then, if* KV *is the tangent to* KKK, *meeting the axis in* V, $1/PV = 1/PR - 1/PT$; *or* PV *is a fourth proportional to* $PT - PR$, PR, *and* PT.

The elegance of the geometrical results probably accounts for the fact that Barrow adheres to the subtangent, as used by Descartes, Fermat, and others; and this would tend to keep from him the further discoveries and development that awaited the man who considered, instead of the subtangent, the much more fertile idea of the gradient, as represented by Leibniz' later development, dy/dx; the germ of the idea of the gradient is of course contained in the "*a* and *e*" method, but it is neglected.

Note the disappearance of the constant Z; hence the curves may be drawn to any convenient scale, which need not be the same, for all or any, in the direction parallel to PEG. The analytical equivalents are :—

(i) If $w = yz$, then $(1/w)dw/dx = (1/y)dy/dx + (1/z)dz/dx$;

(ii) if $v = y/z$, then $(1/v)dv/dx = (1/y)dy/dx - (1/z)dz/dx$.

The first of these results is generally given in modern text-books on the calculus, but I do not remember seeing the second in any book. Thus, for products and quotients we may state the one rule :—

$$\text{If } y = \frac{uv}{wz},\ \frac{dy}{dx} = \frac{uv}{wz}\left[\frac{1}{u}\frac{du}{dx} + \frac{1}{v} \cdot \frac{dv}{dx} - \frac{1}{w} \cdot \frac{dw}{dx} - \frac{1}{z}\frac{dz}{dx}\right]$$

where u, v, w, z, and y are all functions of x.

LECTURE X

Rigorous determination of ds/dx. Differentiation as the inverse of integration. Explanation of the "Differential Triangle" method; with examples. Differentiation of a trigonometrical function.

1. Let AEG be any curve whatever, and AFI another curve so related to it that, if any straight line EF is drawn parallel to a straight line given in position (which cuts AEG in E and AFI in F), EF is always equal to the arc AE of the curve AEG, measured from A; also let the straight line ET touch the curve AEG at E, and let ET be equal to the arc AE; join TF; then TF touches the curve AFI.

2. Moreover, if the straight line EF always bears any the same ratio to the arc AE, in just the same way FT can be shown to touch the curve AFI.*

3. Let AGE be any curve, D a fixed point, and AIF be another curve such that, if any straight line DEF is drawn through D, the intercept EF is always equal to the arc AE; and let the straight line ET touch the curve AGE; make

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* Since the arc is a function of the ordinate, this is a special case of the differentiation of a sum, Lect. IX, § 12; it is equivalent to $d(as+y)/dx = a \,.\, ds/dx + dy/dx$; see note to § 5.

TE equal to the arc AE *; let TKF be a curve such that, if any straight line DHK is drawn through D, cutting the curve TKF in K and the straight line TE in H, HK = HT; then let FS be drawn † to touch TKF at F; FS will touch the curve AIF also.

4. Moreover, if the straight line EF is given to bear any the same ratio to the arc AE, the tangent to it can easily be found from the above and Lect. VIII, § 8.

5. Let a straight line AP and two curves AEG, AFI be so related that, if any straight line DEF is drawn (parallel to AB, a straight line given in position), cutting AP, AEG, AFI, in the points D, E, F respectively, DF is always equal to the arc AE; also let ET touch the curve AEG at E; take TE equal to the arc AE, and draw TR parallel to AB to cut AP in R; then, if RF is joined, RF touches the curve AFI.

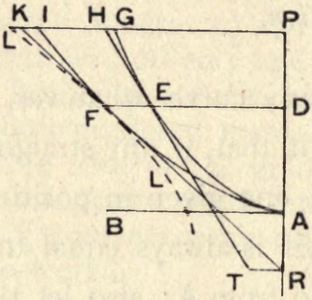

Fig. 106.

For, assume that LFL is a curve such that, if any straight line PL is drawn parallel to AB, cutting AEG in G, TE in H, and LFL in L, the straight line PL is always equal to TH and HG taken together. Then PL > arc AEG > PI; and therefore the curve LFL touches the curve AFI. Again, by Lect VI, § 26, PK = TH (or KL = GH); hence the curve LFL touches the line RFK (by Lect. VII, § 3); therefore the line RFK touches the curve AFI.*

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* TE, AE are drawn in the same sense.

† By Lect. VIII, § 19.

6. Also, if DF always bears any the same ratio to the arc AE, RF will still touch the curve AFI; as is easily shown from the above and Lect. VIII, § 6.

7. Let a point D and two curves AGE, DFI be so related that, if any straight line DFE is drawn through D, the straight line DF is always equal to the arc AE; also let the straight line ET touch the curve AGE at E; make ET equal to the arc AE; and assume that DKK is a curve such that, if any straight line DH is drawn through D, cutting DKK in K and TE in H, the straight line DK is always equal to TH. Then, if FS is drawn (by Lect. VIII, § 16) to touch the curve DKK at F, FS touches the curve DIF also.

8. Moreover, if DF always bears any the same ratio to the arc AE, the straight line touching the curve DIF can likewise be drawn; and in every case the tangent is parallel to FS.

9. By this method can be drawn not only the tangent to the Circular Spiral, but also the tangents to innumerable other curves produced in a similar manner.

10. Let AEH be a given curve, AD any given straight line

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* The proof of this theorem is given in full, since not only is it a fine example of Barrow's method, but also it is a *rigorous* demonstration of the principle of fluxions, that the motion along the path is the resultant of the two rectilinear motions producing it. Otherwise, for rectangular axes, $(ds/dx)^2 = 1 + (dy/dx)^2$; for $ds/dx =$ DF/DR = ET/DR = *Cosec* DET and $dy/dx =$ *Cot* DET.

in which there is a fixed point D, and DH a straight line given in position; also let AGB be a curve such that, if any point G is taken in it, and through G and D a straight line is drawn to cut the curve AEH in E, and GF is drawn parallel to DH to cut AD in F, the arc AE bears to AF a given ratio, X to Y say; also let ET touch the curve AEH; along ET take EV equal to the arc AE; let OGO be a curve such that, if any straight line DOL is drawn, cutting the curve OGO in O and ET in L, and if OQ is drawn parallel to GF, meeting AD in Q, LV : AQ = X : Y. Then the curve OGO is a hyperbola (as has been shown).* Then, if GS touches this curve, GS will touch the curve AGB also.

If the curve AEH is a quadrant of a circle, whose centre is D, the curve AGB will be the ordinary Quadratrix. Hence the tangent to this curve (together with tangents to all curves produced in a similar way) can be drawn by this method.

I meant to insert here several instances of this kind; but really I think these are sufficient to indicate the method, by which, without the labour of calculation, one can find tangents to curves and at the same time prove the constructions. Nevertheless, I add one or two theorems, which it will be seen are of great generality, and not lightly to be passed over.

11. Let ZGE be any curve of which the axis is AD; and let ordinates applied to this axis, AZ, PG, DE, continually

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* Only proved for a special case in Lect. VI, § 17; but the method can be generalized without difficulty.

increase from the initial ordinate AZ; also let AIF be a line such that, if any straight line EDF is drawn perpendicular to AD, cutting the curves in the points E, F, and AD in D, the rectangle contained by DF and a given length R is equal to the intercepted space ADEZ; also let DE : DF = R : DT, and join DT. Then TF will touch the curve AIF.

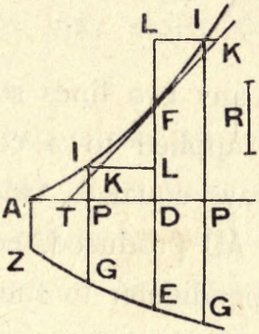

Fig. 109.

For, if any point I is taken in the line AIF (first on the side of F towards A), and if through it IG is drawn parallel to AZ, and KL is parallel to AD, cutting the given lines as shown in the figure; then LF : LK = DF : DT = DE : R, or R . LF = LK . DE.

But, from the stated nature of the lines DF, PK, we have R . LF = area PDEG; therefore LK . DE = area PDEG < DP . DE; hence LK < DP < LI.

Again, if the point I is taken on the other side of F, and the same construction is made as before, plainly it can be easily shown that LK > DP > LI.

From which it is quite clear that the whole of the line TKFK lies within or below the curve AIFI.

Other things remaining the same, if the ordinates, AZ, PG, DE, continually decrease, the same conclusion is attained by similar argument; only one distinction occurs, namely, in this case, contrary to the other, the curve AIFI is concave to the axis AD.

COR. It should be noted that DE . DT = R . DF = area ADEZ.*

12. From the preceding we can deduce the following theorem.

Let ZGE, AKF be any two lines so related that, if any straight line EDF is applied to a common axis AD, the square on DF is always equal to twice the space ADEZ; also take DQ, along AD produced, equal to DE, and join FQ; then FQ is perpendicular to the curve AKF.

I will also add the following kindred theorems.

13. Let AGEZ be any curve, and D a certain fixed point such that the radii, DA, DG, DE, drawn from D, decrease continually from the initial radius DA; then let DKE be another curve intersecting the first in E and such that, if any straight line DKG is drawn through D, cutting the curve AEZ in G and the curve DKE in K, the rectangle contained by DK and a given length R is equal to the area ADG; also let DT be drawn perpendicular to DE, so that DT = 2R; join TE. Then TE touches the curve DKE.

Moreover, if any point, K say, is taken in the curve DKE, and through it DKG is drawn, and DG : DK = R : P; then, if DT is taken equal to 2P and TG is joined, and also KS is drawn parallel to GT; KS will touch the curve DKE.

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* See note at end of this lecture.

Observe that Sq. on DG : Sq. on DK = 2R : DS.

Now, the above theorem is true, and can be proved in a similar way, even if the radii drawn from D, DA, DG, DE, are equal (in which case the curve AGEZ is a circle and the curve DKE is the Spiral of Archimedes), or if they continually increase from A.

14. From this we may easily deduce the following theorem.

Let AGE, DKE be two curves so related that, if straight lines DA, DG are drawn from some fixed point D in the curve DKE (of which the latter cuts the curve DKE in K), the square on DK is equal to four times the area ADG ; draw DH perpendicular to DG, and make DK : DG = DG : DH ; join HK ; then HK is perpendicular to the curve DKE.

We have now finished in some fashion the first part, as we declared, of our subject. Supplementary to this we add, in the form of appendices, a method for finding tangents by calculation frequently used by us (*a nobis usitatum*). Although I hardly know, after so many well-known and well-worn methods of the kind above, whether there is any advantage in doing so. Yet I do so on the *advice of a friend*; and all the more willingly, because it seems to be more profitable and general than those which I have discussed.*

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* See note at the end of this lecture.

Let AP, PM be two straight lines given in position, of which PM cuts a given curve in M, and let MT be supposed to touch the curve at M, and to cut the straight line at T.

In order to find the quantity of the straight line PT,* I set off an indefinitely small arc, MN, of the curve; then I draw NQ, NR parallel to MP, AP; I call MP $= m$, PT $= t$, MR $= a$, NR $= e$, and other straight lines, determined by the special nature of the curve, useful for the matter in hand, I also designate by name; also I compare MR, NR (and through them, MP, PT) with one another by means of an equation obtained by calculation; meantime observing the following rules.

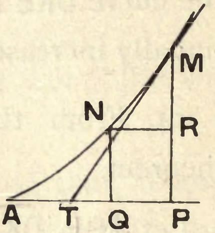

Fig. 115.

Rule 1. In the calculation, I omit all terms containing a power of a or e, or products of these (for these terms have no value).

Rule 2. After the equation has been formed, I reject all terms consisting of letters denoting known or determined quantities, or terms which do not contain a or e (for these terms, brought over to one side of the equation, will always be equal to zero).

Rule 3. I substitute m (or MP) for a, and t (or PT) for e. Hence at length the quantity of PT is found.

Moreover, if any indefinitely small arc of the curve enters the calculation, an indefinitely small part of the tangent, or of any straight line equivalent to it (on account of the indefinitely small size of the arc) is substituted for the arc. But these points will be made clearer by the following examples.

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* See note at the end of this lecture.

NOTE

Barrow gives five examples of this, the "differential triangle" method. As might be expected, two of these are well-known curves, namely the Folium of Descartes, called by Barrow *La Galande*, and the Quadratrix; a third is the general case of the quasi-circular curves $x^n + y^n = a^n$; the fourth and fifth are the allied curves $r = a \, . \, tan \, \theta$ and $y = a \, . \, tan \, x$. It is noteworthy, in connection with my suggestion that Barrow used calculus methods to obtain his geometrical constructions, that he has already given a purely geometrical construction for the curve $r = a \, . \, tan \, \theta$ in Lect. VIII, § 18, if the given lines are supposed to be at right angles. I believe that Barrow, by including this example, intends to give a hint as to how he made out his geometrical construction: thus:—

The equation of the curve is $x^4 + x^2y^2 = a^2y^2$; the gradient, as he shows is $x(2x^2 + y^2)/y(a^2 - x^2)$; using the general letters x and y instead of his p and m. Descartes has shown that a hyperbola is a curve having an equation of the second degree, hence Barrow knows that its gradient is the quotient of two linear expressions, and finds (? by equating coefficients) the hyperbola whose gradient is $x_0(2xx_0 + yy_0)/y_0(a^2 - xx_0)$; the feasibility of this is greatly enhanced by the fact that Barrow would have written the two gradients as

$$\begin{aligned} m : t = {} & x(2xx + yy) : y(aa - xx) \text{ and} \\ & x_0(2xx_0 + yy_0) : y_0(aa = xx_0). \end{aligned}$$

These two gradients are the same at the point x_0, y_0; hence if he can find such a hyperbola, it will touch the curve; he can draw its tangent, and this will also be a tangent to his curve. The curve does turn out to be a hyperbola; for its equation is $x^2x^2{}_0 + x_0y_0 \, . \, xy = a^2yy_0$ or $x^2 + y^2 = y[y + (y_0/x_0) \, . \, (d - x)]$, where $d = a^2/x$. This latter form is easily seen to be equivalent to the construction, in Lect. VIII, § 17, for the curve DYO, when the axes are rectangular; for the equation gives $DY^2 = YE . YR$. It also suggests that the construction for the original curve is transformable into that of § 17, as is proved by Barrow in § 18, and in order that Barrow may draw the tangent, §§ 10, 11, 12 of Lect. VI are necessary to prove that the auxiliary is a hyperbola of which the asymptotes can be determined by a fairly easy geometrical construction. Barrow then generalizes his theorems for oblique axes. I contend that this suggestion is a very probable one for three reasons: (i) it is quite feasible, even if it is considered to be far-fetched, (ii) we know that mathematicians of this time were jealous of their methods, and gave cryptogrammatic hints only in their work (*cf.* Newton's anagram), and (iii) it is to my mind the only reason why this particular theorem should have been selected (especially as Barrow makes it his Example 1), for there is no great intrinsic worth in it.

The fifth example, the case of the curve $y = a . \tan \theta$, I have selected for giving in full, for several reasons. It is the clearest and least tedious example of the method, it is illustrated by two diagrams, one being derived from the other, and therefore the demonstration is less confused, it is connected with the one discussed above and suggests that Barrow was aware of the analogy of the differential form of the polar subtangent with the Cartesian subtangent, and that in this is to be found the reason why Barrow gives, as a rule, the polar forms of all his Cartesian theorems; and lastly, and more particularly, for its own intrinsic merits, as stated below. Barrow's enunciation and proof are as follows:—

EXAMPLE 5. Let DEB be a quadrant of a circle, to which BX is a tangent; then let the line AMO be such that, if in the straight line AV any part AP is taken equal to the arc BE, and PM is erected perpendicular to AV, then PM is equal to BG the tangent of the arc BE.

Take the arc BF equal to AQ and draw CFH; drop EK, FL perpendicular to CB. Let CB $= r$, CK $= f$, KE $= g$.

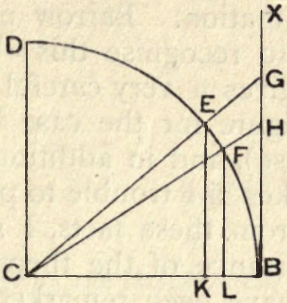


Fig. 120.

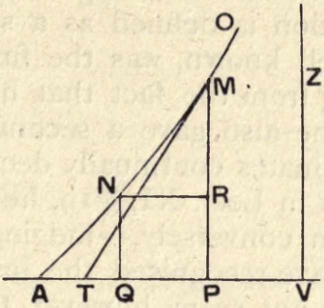


Fig. 121.

Then, since CE : EK = arc EF : LK = QP : LK; therefore $r : g = e :$ LK, or LK $= ge/r$, and CL $= f+ge/r$; hence also LF $= \surd(r^2-f^2-2fge/r) = \surd(g^2-2fge/r)$.

But CL : LF = CB : BH, or $f+ge/r : \surd(g^2-2fge/r) = r : m-a$; and squaring, we have

$$f^2+2fge/r : g^2-2fge/r = r^2 : m^2-2ma.$$

Hence, omitting the proper terms, we obtain the equation

$$rfma = gr^2e+gm^2e;$$

and, on substituting m, t for a, e, we get

$$rfm^2 = gr^2t+gm^2t, \quad \text{or} \quad rfm^2/(gr^2+gm^2) = t.$$

Hence, since $m = rg/f$, we obtain

$$t = m \,.\, r^2/(r^2+m^2) = \mathrm{BG}\,.\,\mathrm{CB}^2/\mathrm{CG}^2 = \mathrm{BG}\,.\,\mathrm{CK}^2/\mathrm{CE}^2.$$

In other words, this theorem states that, if $y = tan\ x$, where x is the circular measure of an "angle" or an "arc," then $dy/dx = m/t = \mathrm{CE}^2/\mathrm{CK}^2 = sec^2\ x$.

Moreover, although Barrow does not mention the fact, he must have known (for it is so self-evident) that the same two diagrams can be used for any of the trigonometrical ratios. Therefore *Barrow must be credited with the differentiation of the circular functions.* (See Note to § 15 of App. 2 of Lect. XII.)

As regards this lecture, it only remains to remark on the fact that the theorem of § 11 is a rigorous proof that differentiation and integration are inverse operations, where integration is defined as a summation. Barrow not only, as is well known, was the first to recognise this; but also, judging from the fact that he gives a very careful and full proof (he also gave a second figure for the case in which the ordinates continually decrease), and in addition, as will be seen in Lect. XI, § 19, he takes the trouble to prove the theorem conversely,—judging from these facts, I say,—he must have recognised the importance of the theorem also. It does not seem, however, to have been remarked that he ever made any use of this theorem. He, however, does use it to prove formulæ for the centre of gravity and the area of a paraboliform, which formulæ he only quotes with the remark, "of which the proofs may be deduced in various ways from what has already been shown, without much difficulty" (see note to Lect. XI, § 2).

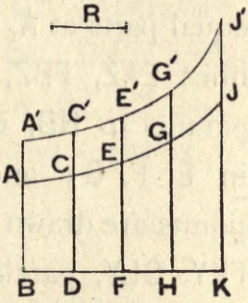
The "differential triangle" method has already been referred to in the Introduction; it only remains to point out the significance of certain words and phrases. Barrow, whilst he acknowledges that the method "seems to be more profitable and *more general* than those which I have discussed," yet is in some doubt as to the advantage of including it, and almost apologizes for its insertion; probably, as I suggested, because, although he has found it a most useful tool for hinting at possible geometrical constructions, yet he compares it unfavourably as a method with the methods of pure geometry. It is also to be observed that his axes are not necessarily rectangular, although in the case of oblique axes, PT can hardly be accepted as the subtangent; hence he finds it convenient to tacitly assume that his axes are at right angles. The last point is that Barrow distinctly states that his method is expressly "in order to find the quantity of the subtangent," and I consider that this is almost tantamount to a direct assertion that he has used it frequently to get his first hint for a construction in one of his problems. The final significance of the method is that by it he can readily handle *implicit* functions.

LECTURE XI

Change of the independent variable in integration. Integration the inverse of differentiation. Differentiation of a quotient. Area and centre of gravity of a paraboliform. Limits for the arc of a circle and a hyperbola. Estimation of π.

NOTE

In the following theorems, Barrow uses his variation of the usual method of summation for the determination of an area. If ABKJ is the area under the curve AJ, he divides BK into an infinite number of equal parts and erects ordinates. In his figures he generally makes four parts do duty for the infinite number.



He then uses the notation already mentioned, namely, that the area ABKJ is equal to the sum of the ordinates AB, CD, EF, GH, JK.

The same idea is involved when he speaks of the sum of the rectangles CD . DB, EF . FD, GH . FH, JK . GH; for this sum, where commas are used between the quantities instead of a plus sign, does not stand for the area ABKJ, but for R . A′BKJ′, where an ordinate HG′ is such that R . HG′ = HG . FH, and R is some given length; in other words, ordinates proportional to each of the rectangles are applied to points of the line BK, and their aggregate or sum is found; hence this sum is of three dimensions. On the contrary, he uses the same phrase, with plus signs instead of commas, to stand for a simple summation.

Thus, in § 3 of this lecture, the sum of $AZ . AE^2$, $BZ . BF^2$, $CZ . CG^2$, etc., is the area aggregated from ordinates proportional to $AZ . AE^2$, $BZ . BF^2$, $CZ . CG^2$, etc., applied to the line VD; and it is of the fourth dimension. Whereas, in § 3, the sum $HL . HO^2 + LK . LY^2 + KI . KY^2 +$ etc., is aggregated from ordinates equal to HO^2, LY^2, KY^2, etc., applied to the line HD; and it is the same as the sum of HO^2, LY^2, KY^2, etc.

1. If VH is a curve whose axis is VD, and HD is an ordinate perpendicular to VD, and $\phi Z \psi$ is a line such that, if from any point chosen at random on the curve, E say, a straight line EP is drawn normal to the curve, and a straight line EAZ perpendicular to the axis, AZ is equal to the intercept AP; then the area $VD\psi\phi$ will be equal to half the square on the line DH.

For if the angle HDO is half a right angle, and the straight line VD is divided into an infinite number of equal parts at A, B, C, and if through these points straight lines EAZ, FBZ, GCZ, are drawn parallel to HD, meeting the curve in E, F, G; and if from these points are drawn straight lines EIY, FKY, GLY, parallel to VD or HO; and if also EP, FP, GP, HP are normals to the curve, the lines intersecting as in the figure; then the triangle HLG is similar to the triangle PDH (for, on account of the infinite section, the small arc HG can be considered as a straight line).

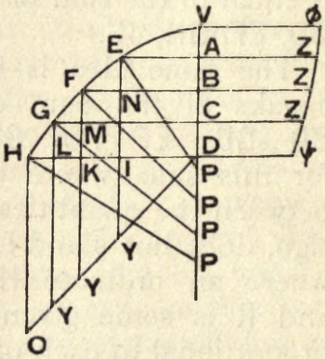


Fig. 122.

Hence, $\mathrm{HL : LG = PD : DH}$, or $\mathrm{HL . DH = LG . PD}$,

i.e. $\mathrm{HL . HO = DC . D\psi}$.

By similar reasoning it may be shown that, since the triangle GMF is similar to the triangle PCG, $\mathrm{LK . LY = CB . CZ}$; and in the same way, $\mathrm{KI . KY = BA . BZ}$, $\mathrm{ID . IY = AV . AZ}$.

Hence it follows that the triangle DHO (which differs in the slightest degree only from the sum of the rectangles $\mathrm{HL . HO + LK . LY + KI . KY + ID . IY}$) is equal to the space $\mathrm{VD\psi\phi}$ (which similarly differs in the least degree only from the sum of the rectangles $\mathrm{DC . D\psi + CB . CZ + BA . BZ + AV . AZ}$);

i.e. $\mathrm{DH^2/2}$ = space $\mathrm{VD\psi\phi}$.

A lengthier indirect argument may be used; but what advantage is there?

2. With the same data and construction as before, the sum of the rectangles $\mathrm{AZ . AE}$, $\mathrm{BZ . BF}$, $\mathrm{CZ . CG}$, etc., is equal to one-third of the cube on the base DH.

For, since $\mathrm{HL : LG = PD : DH = PD . DH : DH^2}$; therefore $\mathrm{HL . DH^2 = LG . PD . DH}$ or $\mathrm{LH . HO^2 = DC . D\psi . DH}$; and, similarly $\mathrm{LK . LY^2 = CB . CZ . CG}$, $\mathrm{KI . KY^2 = BA . BZ . BF}$, etc.

But the sum $\mathrm{HL . HO^2 + LK . LY^2 + KI . KY^2} +$ etc. $= \mathrm{DH^3/3}$;* and the proposition follows at once.

3. By similar reasoning, it follows that

the sum of $\mathrm{AZ\ AE^2}$, $\mathrm{BZ . BF^2}$, $\mathrm{CZ . CG^2}$, etc. $= \mathrm{DH^4/4}$;

the sum of $\mathrm{AZ . AE^3}$, $\mathrm{BZ . BF^3}$, $\mathrm{CZ . CG^3}$, etc. $= \mathrm{DH^5/5}$;

and so on.†

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* See the critical note immediately following.

† The analytical equivalents of the theorems given above are comprised in the general formula (with their proofs),

$$\int y^r (dy/dx) . dx = \int y^r . dy = y^{r+1}/(r+1).$$

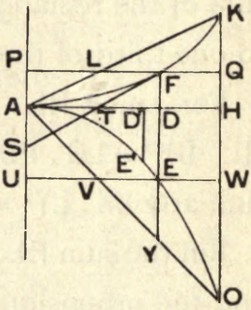
NOTE

On the assumptions in the proofs of §§ 2, 3.

The summation used in § 2 has already been given by Barrow in Lect. IV; he states that it has been established "in another place" (? by Wallis or others), and that it at least "is sufficiently known among *geometers*."

It is easy, however, to give a demonstration according to Barrow's methods of the general case; and, since in several cases Barrow is content with saying that the proof may easily be obtained by his method, and sometimes he adds "in several different ways," I feel sure that he had made out a proof for these summations in the general case.

The method given below follows the idea of Lect. IV, by *finding a curve convenient for* the summation, without proving that this curve is the only one that will do. Other methods will be given later, thus substantiating Barrow's statement that the matter may be proved in several ways; see notes following Lect. XI, § 27, and Lect. XI, App., § 2.



Let AH, KHO be two straight lines at right angles, and let $AH = HO = R$ and $KH = S$. Let AEO be a curve such that (in the figure) UE is the first of $n-1$ geometrical means between UW and UV,

i.e. $UE^n = UW^{n-1} . UV$,

or $DY^n = AD^n = R^{n-1} . DE$.

Let AFK be a curve such that PF is the first of n geometrical means between PQ and PL, or $PF^{n+1} = PQ^n . PL$, *i.e.* $S . AD^{n+1} = R^{n+1} . DF$.

Then the curve AFK is a curve that is fitted for the determination of the area under the curve AED, providing a suitable value of S/R is chosen.

For $FD/DT = PS/AD = (n+1) . DF'/AD = (n+1) . DE . S/R^2$; and if S is taken equal to $R/(n+1)$, $FD : DT = DE : R$; and therefore by Lect. X, § 11,

$$\begin{aligned} \text{the sum } DY^n . DD' + \ldots &= \text{the sum } R^{n-1} . DE . DD' \\ &= R^{n-1} . \text{area } ADE \\ &= R^n . DF = S . AD^{n+1}/R = DY^{n+1}/(n+1). \end{aligned}$$

Hence we may deduce the following important theorems:—

4. Let $VD\psi\phi$ be any space of which the axis VD is equally divided (as in fig. 122); then if we imagine that each of the spaces $VAZ\phi$, $VBZ\phi$, $VCZ\phi$, etc., is multiplied by its own ordinate AZ, BZ, CZ, etc., respectively, the sum which is produced will be equal to half the square of the space $VD\psi\phi$.

5. If, however, each of the square roots of the spaces is multiplied by its own ordinate, an aggregate is produced equal to two-thirds of the square root of the cube of $VD\psi\phi$.*

6. *Example.*—Let $VD\psi$ be a quadrant of a circle, of which the radius is R and the perimeter is P; then the segments VAZ, VBZ, VCZ, . . ., each multiplied by its own sine, AZ, BZ, CZ, . . ., respectively, will together make $R^2P^2/8$.

Also the sum $AZ\,.\,\surd(VAZ) + BZ\,.\,\surd(VBZ) + \text{etc.}$

$$= \surd(R^3P^3/8)\,.\,2/3 = \surd(R^3P^3/18).$$

Again, if $VD\psi$ is a segment of a parabola, the sum made from the products into the ordinates will be equal to 2/9 of $VD^2\,.\,D\psi^2$, and that from the products of the square roots of the segments into the ordinates will be equal to 2/3 of $\surd(8/27 \text{ of } VD^3\,.\,D\psi^3)$ or $\surd(VD^3\,.\,D\psi^3\,.\,32/243)$.

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* The equivalents of §§ 4, 5, are respectively $\int y(\int y\,.\,dx)\,.\,dx = (\int y\,.\,dx)^2$ and $\int y\surd(\int y\,.\,dx)\,.\,dx = (\int y\,.\,dx)^{3/2}.\ 2/3$; or in a more recognizable form, putting z for $\int y\,.\,dx$, they are $\int z(dz/dx)\,.\,dx = z^2/2$, and so on.

Other similar things concerning the sums made from the products of other powers and roots of the segments into the ordinates or sines can be obtained.

7. Further, it follows from what has gone before that, in every case, if the lines VP intercepted between the vertex and the perpendiculars are supposed to be applied through the respective points A, B, C, . . . , say that AY, BY, CY, . . . are equal to the respective lines VP; then will the space VDξθ, constituted by these applied lines, be equal to half the square on the subtense VH.

8. Moreover, if with the same data, RXXS is a curve such that IX = AP, KX = BP, LX = CP, . . . ; then the solid formed by the rotation of the space VDψφ about VD as an axis is half the solid formed by the rotation of the space DRSH about the same axis VD.

9. All the foregoing theorems are true, and for similar reasons, even if the curve VEH is convex to the line VD.

From these theorems, the dimensions of a truly boundless number of magnitudes (proceeding directly from their construction) may be observed, and easily verified by trial.

10. Again, if VH is a curve, whose axis is VD and base DH, and DZZ is a curve such that, if any point such as E is taken on the curve VH and ET is drawn to touch the curve, and a straight line EIZ is drawn parallel to

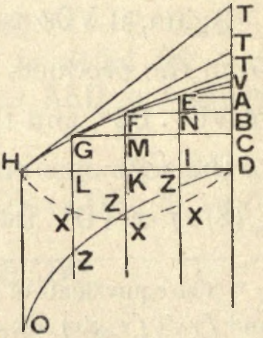

Fig. 125.

the axis, then IZ is always equal to AT; in that case, I say, the space DHO is equal to the space VHD.

This extremely useful theorem is due to that most learned man, Gregory of Aberdeen:* we will add some deductions from it.

11. With the same data, the solid formed by the rotation of the space DHO about the axis VD is twice the solid formed by the rotation of the space VDH about the same axis.

For $HL : LG = DH : DT = DH : HO = DH^2 : DH . HO$;

$$HL . DH . HO = LG . DH^2 = CD . DH^2.$$

Similarly, $LK.DL.LZ = BC.CG^2$, $KI.DK.KZ = AB.BF^2$,

and $ID . DI . IZ = VA . AE^2$.

But it is well known that †

the sum $CD . DH^2 + BC . CG^2 + AB . BF^2 + VA . AE^2$

= twice the sum of $DI . IE$, $DK . KF$, $DL . LG$, etc.,

and therefore the solid formed by the space DHO rotated about the axis VD is double of the solid formed by the space VDH rotated round VD.

12. Hence the sum of $DI . IZ$, $DK . KZ$, $DL . LZ$, etc. (aplied to HD) = the sum of the squares on the ordinates to VD,

= the sum of AE^2, BF^2, CG^2, etc. (applied to VD).

The same even tenor of conclusions is observable for the other powers.

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[*] The member of a remarkable family of mathematicians and scientists that is here referred to is James Gregory (1638–1675), who published at Padua, in 1668, *Geometriæ Pars Universalis.* He also gave a method for infinitely converging series for the areas of the circle and hyperbola in 1667.

[†] For a discussion of this and §§ 12, 13, 14, see the critical note on page 133.

13. By similar reasoning, it follows that

the sum of $DI^2 . IZ$, $DK^2 . KZ$, $DL^2 . LZ$, etc. (applied to HD)
= three times the sum of the cubes on all the ordinates AE, BF, CG, . . ., applied to VD.

14. With the same data, if DXH is a curve such that any ordinate to DH, as IX, is a mean proportional between the ordinates IE, IZ congruent to it; then the solid formed by the space VDH rotated about the axis DH is double the solid formed from the space DXH rotated about the same axis.

15. If, however (in fig. 125), the curve DXH is supposed to be such that any ordinate, CX say, is a bimedian * between the congruent ordinates IE, IZ; then the sum of the cubes of IX, KX, LX, etc., is one-third of the sum of the cubes of DV, IE, KF, etc. But if IX is a trimedian *; then the sum of IX^4, KX^4, LX^4, etc., is equal to one-fourth of the sum of DV^4, IE^4, KF^4, etc.; and so on for all the other powers.

* NOTE. I call by the name *bimedian* the first of two mean proportionals, by *trimedian* the first of three, and so on.

These results are deduced and proved by similar reasoning to that of the previous propositions; but repetition is annoying.†

16. Again, if VYQ is a line such that the ordinate AY is equal to AT, BY to BT, and so on; then the sum of IZ^2, KZ^2, LZ^2, etc., that is, the sum of the squares of the ordinates

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† Literally, "it irks me to cry cuckoo."

of the curve DZO applied to the line DH, is equal to the sum VA . AE . AY + AB . BF . BY + etc., that is, the figure VDH "multiplied" by the figure VDQ.*

17. Also the sum of IZ^3, KZ^3, LZ^3, etc.

= the sum VA . AE . AY^2 + AB . BF . BY^2 + etc. ;*

that is, the figure VDH "multiplied" by the figure VDQ "squared."

These you can easily prove by the pattern of the proofs given above.

18. The same things are true and are proved in an exactly similar manner, even if the curve VH is convex to the straight line VD.

NOTE

The equality, which in § 11 is said by Barrow to be well known, namely, the sum CD . DH^2 + BC . CG^2 + etc.

= twice the sum of DI . IE, DK . KF, etc.,

is really an equality between two expressions for the volume of the solid formed by the rotation of VDH about VD; and the analytical equivalent is $\int y^2\,dx = 2\int xy\,dy$, with the intermediate step $2\int\int y\,dy\,dx$. Thus these theorems of Barrow are equivalent to the equality of the results obtained from a double integral, when the two first integrals are obtained by integrating with regard to each of the two variables in turn. He says indeed that the first result, that in § 11, is a matter of common knowledge, but he remarks that the others that he uses in the following sections can be obtained by similar reasoning. From this, and from indications in Lect. IV, § 16, Lect. X, § 11, and § 19 of this lecture, I feel certain that Barrow had obtained these theorems in the course of his researches, but, as in many other cases, he omits the proofs and leaves them to the reader. All the more, because the proofs follow very easily by his methods. Take, for the sake of example, the case of the equivalent of $\int\int y^3\,dy\,dx$, for which I imagine Barrow's method would have been somewhat as follows.

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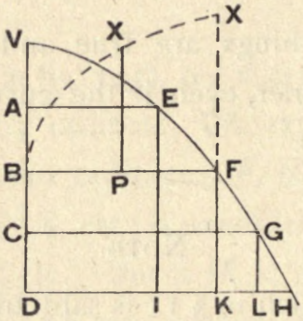
* The equivalents of these theorems are:

16. $\int[y/(dy/dx)]^2.dy = \int y\,.\,[y/(dy/dx)]\,.\,dx.$

17. $\int[y/(dy/dx)]^3.dy = \int y\,[y/(dy/dx)]^2\,.\,dx.$

In order to find the aggregate of all the points of a given space VDH, each multiplied by the cube of its distance from the axis VD, we may proceed in two different ways.

Method 1.—By applying lines PX, proportional to the cube of BP, to every point P of the line BF, find the aggregate of the products for the line BF; then find the sum of these aggregates for all the lines applied to VD. Here, since PX is proportional to BP^3, the curve BXX is a cubical parabola, and the space BFX is one-fourth of the fourth power of BF; and Barrow would write the result of the summation as the sum of $AE^4/4$, $BF^4/4$, $CG^4/4$, etc. (applied to VD).



Method 2.—Find the aggregates along all the parallels to VD, and then the sum of these aggregates applied to DH. Here the first aggregates are represented by rectangles whose bases are IE, KF, etc., and whose heights are equal to DI^3, DK^3, etc., respectively; and Barrow would write this as the sum of $DI^3 . IE$, $DK^3 . KF$, etc. (applied to DH).

Lastly, in fig. 125, $ID . DI^3 . IZ = VA . AE^3 . DI$, etc.; hence all the results follow immediately; *i.e.* $\int\int y^3\,dx\,dy$ is equal to either of the integrals $\int y^4/4\,dx$, $\int xy^3\,dy$; and Barrow proves that these are equal to one-fourth of $\int y^3 y/(dy/dx)\,dy$.

19. Again, let AMB be a curve of which the axis is AD and let BD be perpendicular to AD; also let KZL be another line such that, when any point M is taken in the curve AB, and through it are drawn MT a tangent to the curve AB, and MFZ parallel to DB, cutting KZ in Z and AD in F, and R is a line of given length, TF : FM = R : FZ. Then the space ADLK is equal to the rectangle contained by R and DB.*

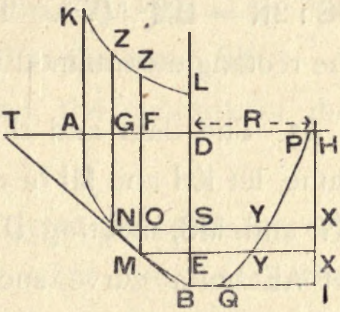


Fig. 127.

For, if DH = R and the rectangle BDHI is completed, and MN is taken to be an indefinitely small arc of the curve AB, and MEX, NOS are drawn parallel to AD; then we have

NO : MO = TF : FM = R : FZ;

∴ NO . FZ = MO . R, and FG . FZ = ES . EX.

Hence, since the sum of such rectangles as FG . FZ differs only in the least degree from the space ADLK, and the rectangles ES . EX form the rectangle DHIB, the theorem is quite obvious.

20. With the same data, if the curve PYQ is such that the ordinate EY along any line MX is equal to the corresponding FZ; then the sum of the squares on FZ (applied to the line AD) is equal to the product of R and the space DPQB.

21. Similarly, the sum of the cubes of FZ, applied to AD, is equal to the product of R and the sum of the squares

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* This is the converse of Lect. X, § 11.

of EY, applied to BD; and so on in similar fashion for the other powers.

22. Let DOK be any curve, D a fixed point in it, and DKE a chord; also let AFE be a curve such that, when any straight line DMF is drawn cutting the curves in M and F, DS is drawn perpendicular to DM, MS is the tangent to the curve DOK, cutting DS in S, and R is any given straight line, then $DS : 2R = DM^2 : DF^2$. Then the space ADE will be equal to the rectangle contained by R and DK.*

23. The data and the construction being otherwise the same, let KH and MI be drawn perpendicular to the tangents KT and MS, meeting DT, DS in H and I respectively; and let AE be a curve such that $DE = \surd(DK . DH)$, and also $DF = \surd(DM . DI)$, and so on. Then the space ADE is equal to one-fourth of the square on DK.

24. If DOK is any curve, D a given point on it, and DK any chord; also if DZI is a curve such that, when any point M is taken in the curve DOK, DM is joined, DS is drawn perpendicular to DM, MS is a tangent to the curve, DP is taken along DK equal to DM, and PZ is drawn perpendicular to DK, then PZ is equal to DS; in this case the space DZI is equal to twice the space DKOD.†

25. The data and the construction being in other respects the same, let the ordinates PZ now be supposed to be equal

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* The analytical equivalents are:

22. $\int r_1^2 . d\theta = 2R . \int r^2/(r^2 . d\theta/dr) . d\theta = 2Rr.$

23. $\int r_1^2/2 . d\theta = \int r/2 . (dr/d\theta) . d\theta = r^2/4.$

24. $\int r^2 . d\theta = \int r^2 . (d\theta/dr) . dr.$

† See note on page 138.

to the respective tangents MS; take any straight line *xk*, and distances along it equal to the arcs DOK, DOM, DON, etc., and draw the ordinates *kd*, *md*, *nd*, etc., equal to the chords KD, MD, ND, etc.; then the space *xkd* will be equal to the space DKI.

26. Moreover if, other things remaining the same, any straight line *kg* is taken, the rectangle *xkgh* is completed, and the curve DZI is supposed to be such that MD : DS = *kg* : PZ; then the rectangle *xkgh* will be equal to the space DKI.

Hence, if the space DKI is known, the quantity of the curve DOK may be found.

Should anyone explore and investigate this mine, he will find very many things of this kind. Let him do so who must, or if it pleases him.

Perhaps at some time or other the following theorem, too, deduced from what has gone before, will be of service; it has been so to me repeatedly.

27. Let VEH be any curve, whose axis is VD and base DH, and let any straight line ET touch it; draw EA parallel to HD. Also let GZZ be another curve such that, when any straight line EZ is drawn from E parallel to VD, cutting the base HD in I and the curve GZZ in Z, and a straight line of given length R is taken, then at all times $DA^2 : R^2 = DT : IZ$. Then $DA : AE = R^2$: space DIZG; (or, if DA : R is made equal to R : DP, and PQ is drawn parallel to DH, then the rectangle DPQI is equal to the space DGZI). [Fig. 131, p. 139.]

The following theorem is also added for future use.

28. Let AMB be any curve whose axis is AD; also let the

line **KZL** be such that, if any point **M** is taken in **AMB**, and from it are drawn a straight line **MP** perpendicular to **AB** cutting **AD** in **P**, and a straight line **MG** perpendicular to **AD** cutting the curve **KZL** in **Z**, at all times **GM** : **PM** is equal to arc **AM** : **GZ**; then the space **ADKL** will be equal to half the square on the arc **AM**.

These theorems, I say, may be obtained from what has gone before without much difficulty; indeed, it is sufficient to mention them; and, in fact, I intend here to stop for a while.

NOTE

The theorems of §§ 24–28 deserve a little special notice. The first of these was probably devised by Barrow for the quadrature of the Spiral of Archimedes; it included, as was usual with him, "innumerable spirals of other kinds," thus representing both, as Barrow would consider it, an improvement and a generalization of Wallis' theorems on this spiral in the "Arithmetic of Infinites."

It is readily seen that if **DZI** is a straight line, the curve **AOK** is the first branch or turn of the Logarithmic or Equiangular Spiral; if **DZI** is a parabola, the curve **DOK** is the Circular Spiral or Spiral of Archimedes; and if the curve is any paraboliform, the curve **DOK** is a spiral whose equation may be $r^m = A\theta^n$. In short, Barrow has given a general theorem to find the polar area of any curve whose equation is $\theta = \int f(r)/r^2 . dr$, for all cases in which he can find the area under the curve $y = f(x)$.

The theorem of § 26 is indeed remarkable, in that it is a general theorem on *rectification*. It is stated * that Wallis had shown, in 1659, that certain curves were capable of rectification, that William Neil, in 1660, had rectified the *semi-cubical parabola*, using *Wallis' method*, that the second curve to be rectified was the *cycloid*, and that this was effected by Sir C. Wren in 1673. Barrow's general theorem includes as a special case, when the line DZI is a straight line, whose equation is $y = \sqrt{2} . x$, the curve DOK with the relation $ds/dr = \sqrt{2} . r$, that is the triangle DMS is always a right-angled isosceles triangle, and therefore the curve is the Logarithmic or Equiangular Spiral, which may thus be considered to be the real second curve that was rectified. Even if not so, we shall find later that Barrow has anticipated Wren in rectifying the cycloid, as a particular case of another general theorem; and in this case, he distinctly remarks on the fact that he has done so. In general, Barrow's theorem rectifies any curve whose equation is $\theta = \int \sqrt{(R^2 - r^2)/r^2} . dr$, where $R = f(r)$, so long as he can find the area under the curve $y = f(x)$.

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* *Ency. Brit.* (*Times* edition), Art. on *Infinitesimal Calculus*. (Williamson). These dates are wrong, however, according to other authorities, such as Rouse Ball.

The theorem of § 27 is even more remarkable, not only for the value of its equivalent, which is the differentiation of a *quotient*, but also because it is a noteworthy example of what I call Barrow's contributory negligence; for although he recognizes its value, and indeed states that it has been of service to him "*repeatedly*" (and no wonder), yet he thinks that "it is enough to mention it," and omits the proof, which "may be obtained from what has gone before without much difficulty." Even the figure he gives is the worst possible to show the connection, as it involves the consideration that the gradient is negative when the angle of slope is obtuse. Of the figures below, the one on the right-hand side is that given by Barrow;

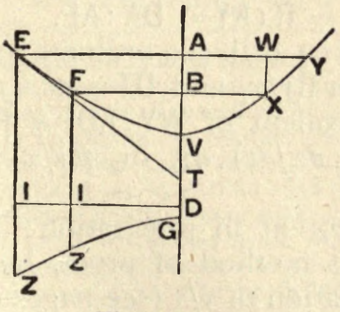


Fig. 131A.

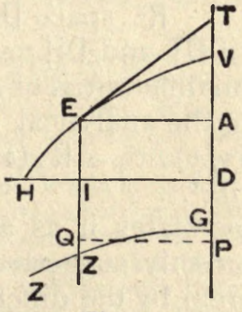
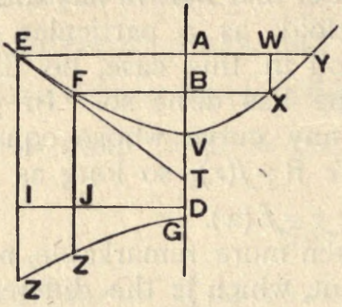
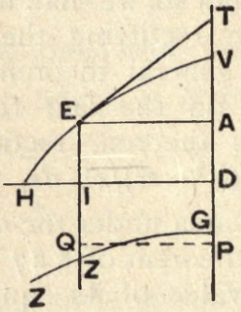


Fig. 131.

the proof, which Barrow omits, may be given as follows, reference being made to the figure on the left-hand side.

Let the curve VXY be such that, if EA produced meets it in Y, then always $EA : AD = AY : R$. Divide the arc EV into an infinite number of parts at F, L, . . . , and draw FBX, LCX, . . . , parallel to HD, meeting VD in B, C, . . . , and the curve VXY in the points X; also draw FJZ, . . . , parallel to VD, meeting HD in J, . . . , and the curve GZZ in the points Z.





Then $AY . AD . BD = R . EA . BD = R . (EA . AD - EA . AB)$,
and $BX . AD . BD = R . FB . AD = R . (EA . AD - IJ . AD)$;
hence, if XW is drawn parallel to VD, cutting AY in W, then

$$WY . AD^2 = WY . AD . BD = R . (IJ . AD - EA . AB).$$

But $EA : AT = IJ : AB$, or $EA . AB = IJ . AT$;

$$WY . AD^2 = R . (IJ . AD - IJ . AT) = R . IJ . DT.$$

Now $DA^2 : R^2 = DT : IZ = IJ . DT : IJ . IZ$;

$$R^2 : IJ . IZ = AD^2 : IJ . DT = R : WY.$$

Hence, since the sum of the rectangles $IJ . IZ$ only differs in the least degree from the space DGZI, and the sum of the lengths WY is AY; it follows immediately that

$$R^2 : \text{space } DGZI = R : AY = DA : AE.$$

Now if DT and DH are taken as the co-ordinate axes, then WY is the differential of AY or Ry/x, and $DT = x - y . dx/dy$; therefore the analytical equivalent of $WY . AD^2 = R . DT . IJ$ is $R . d(y/x) . x^2 = R . (x - y . dx/dy) . dy$, or $d(y/x) = (x . dy - y . dx)/x^2$.

Barrow states it as a theorem in integration; but, if I have correctly suggested his method of proof, he obtains his theorem by the differentiation of y/x (see pages 94, 112).

APPENDIX

1. When *many years ago* I examined the *Cyclometrica* of that illustrious man, Christianus Hugenius,* and studied it closely, I observed that two methods of attack were more especially used by him. In one of these, he showed that the segment of a circle was a mean between two parabolic segments, one inscribed and the other circumscribed, and in this way he found limits to the magnitude of the former. In the other, he showed that the centre of gravity of a circular segment was situated between the centres of gravity of a parabolic segment and a parallelogram of equal altitude, and hence found limits for this point. It occurred to me that in place of the parabola in the first method, and of the parallelogram in the second, some *paraboliform curve* circumscribed to the circular segment could be substituted, so that the matter might be considered somewhat more closely. On examining it, I soon found that this was

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* The work of Christiaan Huygens (1629–1695), the great Dutch mathematician, astronomer, mechanician, and physicist, that is referred to may be the essay *Exetasis quadraturæ circuli* (Leyden, 1651), but more probably is the complete treatise *De circuli magnitudine inventa*, that was published three years later. Putting the date of Barrow's study of Huygens' work at not later than 1656 (note the words in the first line above that I have set in italics—*many years ago*,—and remembering that this was printed in 1670), it follows from Barrow's mention of the paraboliform curve as something well known to him, and from a remark that the proofs of the theorems of § 2 "may be deduced in various ways from what has already been shown, without much difficulty," that Barrow was in possession of his knowledge of the properties of his beloved paraboliforms even before this date. Is it not therefore probable, nay almost certain, that Barrow, in 1655 *at the very latest*, had knowledge of his theorem equivalent to the *differentiation of a fractional power*?

correct; moreover, I easily found that like methods could be used for the magnitude of a hyperbolic segment. As the proofs for these theorems—better perhaps than others that might be invented—are short, and clear (because they follow from or depend on what has been shown above), I thought good to set them forth in this place. I think, too, that they are in other respects not without interest.

2. Let us assume the following as known theorems; of which the proofs may be deduced in various ways from what has already been shown, without much difficulty.

If BAE is a paraboliform curve, whose axis is AD and base or ordinate is BDE, BT a tangent to it, and K the centre of gravity; then, if its exponent is n/m, we have *

$$\text{Area of BAE} = m/(n+m) \text{ of AD . BE},$$
$$\text{TD} = m/n \text{ of AD},$$
$$\text{and} \quad \text{KD} = m/(n+2m) \text{ of AD}.$$

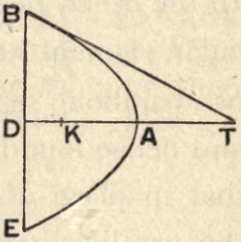

Fig. 133.

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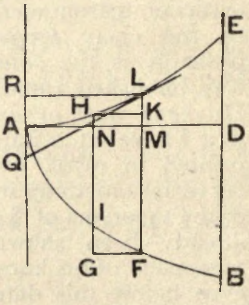
* The definition of Lect. VII, § 12, uses N/M; the value of TD/AD is found in Lect. IX, § 4; where also the definition of the paraboliforms is given.

Now it is clear from the adjoining figure that if AHLE is a paraboliform, whose exponent is $r/s(=1/a$ say), then $\text{LK/HK} = a\,.\,\text{LM/AM}$; and conversely.



Let AIFB be a curve such that

$$\text{FM/R} = \text{LK/HK} = a\,.\,\text{LM/AM};$$

then, by Lect. XI, § 19, area AFBD = R . DE.

But, in this case, we have

$$\text{IG : FM} = \text{LM/AM} - \text{HN/AN : LM/AM}$$
$$= \text{AM . LK} - \text{LM . HK : LM . AN}$$
$$= (a-1)\,.\,\text{LM . HK : LM . AN};$$
$$\text{FG/GI} = 1/(a-1) \text{ of AM/FM}.$$

Hence AIFB is a paraboliform, whose vertex is A, axis AD, exponent $a-1$.

3. Let AEB, AFB be any two curves, having the axis AD and the ordinate BD common, so related that, if any straight line EFG is drawn parallel to BD, cutting the given lines in the points E, F, G, and ES, FT touch the curves AEB, AFB respectively, TG is always greater than SG; then I say that no part of the curve AFB can fall within the curve AEB.

4. Let BAE be any curve, of which AD is the axis, and let the base ADE be an ordinate to it; also let the point H be the centre of gravity of the segment BAE, and RS a straight line through it parallel to BE. Further let another curve (or any line you please) MRASN pass through the points R, S, and have the same axis AD; let it cut the first curve BAE in such a manner that the upper part RKAPS falls within the curve BAE, but the lower remaining parts, RM, SN, fall outside it. Then the centre of gravity of the segment MRASN will be below the point H, that is, towards the base MN.

5. Let the two straight lines BT, ES touch the circle AEB, whose centre is C, and meet the diameter CA in the points T and S; also let the straight lines BD, EP be perpendicular to CA. Then, if AD > AP, TD : AD > SP : AP.

Conversely, if AIFB has an exponent $n/m (= a - 1)$, the integral curve is a paraboliform, exponent $1/a$ or $m/(n+m)$.

Hence, since $DB/R = a . DE/AD$, area $AIFBD = R . DE = m/(n+m)$ of $AD . DB$.

Similarly, area $ALED = AD . DE - (n+m)/(n+2m)$ of $AD . DE$,
$= m/(n+2m)$ of $AD . DE$;

$\therefore$ $R . a$. area ALED : AD . area AIBD $= n+m : n+2m$.

Now, since $FM/R = a . LM/AM$, $\therefore$ $FM . AM . MN = R . a . LM . HK$; hence, summing, AK . area AFBD $= R . a$. area ALED; therefore $AK : AD = n+m : n+2m$, or $KD = m/(n+2m)$ of AD.

In a similar way the centre of inertia could be found.

The proof could have been deduced from the note on § 2 of Lect. XI or by drawing a subsidiary curve as in the note to § 27 of Lect. XI.

6. Let the two straight lines BT, ES now touch a hyperbola AEB, whose centre is C; and let other things be the same as in the theorem just before; then TD : AD $>$ SP : AP.

7. Let the axis AD and the base BD be common to the circle AEB whose centre is C, and the paraboliform AFB; also let the exponent of the paraboliform be n/m, where

$$\text{AD} = (m - 2n)/(m - n) \text{ of CA},$$

or

$$m - n : m - 2n = \text{CA} : \text{AD}.$$

Moreover, let the straight line BT touch the circle; then BT will touch the paraboliform also.

8. It should be noted in this connection that, conversely, if the ratio of AD to CA is given, the paraboliform which touches the circle AEB at B is thereby determined.

For instance, if AD/CA $= s/t$, then $(t - s)/(2t - s)$ will be the exponent of the required paraboliform.

9. With the same hypothesis as in § 7, the paraboliform AFB will fall altogether outside the circle AEB.

10. Again with the same hypothesis, if with a base GE (any parallel to BD) and axis AD another paraboliform is supposed to be drawn, of the same kind as AFB (or having the same exponent n/m); then this curve also, for the part AE above GE, will fall altogether outside the circle.

11. Also it may be shown that the said paraboliform (of like kind to AFB and constructed on the base GE), when produced below GE to DB, will fall altogether within the circle as regards this part.

12. Further, let AD be the axis and DB the base common

to the hyperbola AEB whose centre is C, and the paraboliform AFB, whose exponent is n/m; also let AD = $(2n-m)/(m-n)$ of CA; and let BT touch the hyperbola. Then BT touches the paraboliform also.

13. Hence again, if the ratio of AD to CA is given, the paraboliform touching the hyperbola at the point B is thereby determined. For instance, if AD/CA = s/t, $n/m = (t+s)/(2t+s)$.

14. With the same hypothesis as in § 12, the paraboliform AFB will lie altogether within the hyperbola AEB.

15. Also, with the same hypothesis, if you imagine a paraboliform of the same kind to be constructed with the base GE and axis AG; it will fall within the hyperbola on the upper side of GE.

16. Moreover, if this second like paraboliform, constructed on the base EG, is supposed to be produced to DB; then the part of it intercepted by EG and BD will fall altogether outside the hyperbola.

17. Let the circle AEB and the parabola AFB have a common axis AD and base BD; then the parabola will fall within the circle on the side above BD, and without the circle below BD.

If an ellipse is substituted for the circle, the same result holds and is proved in like manner.

18. Let the hyperbola whose axis is AZ and parameter AH, and the parabola AFB have the same axis AD and base BD; then the parabola will fall altogether outside the

hyperbola above BD, but within it when produced below BD.

19. From what has been said, the following rules for the mensuration of the circle may be obtained.

Let BAE be a part of a circle, of which the axis is AD, and the base BE; let C be the centre of the circle, and EH equal to the right sine of the arc BAE; also let $AD : CA = s : t$.

Then (1) $(2t - s)/(3t - 2s)$ of $AD . BE >$ segment BAE;
(2) $EH + (4t - 2s)/(3t - 2s)$ of $BH >$ arc BAE;
(3) $2/3$ of $AD . BE <$ segment BAE;
(4) $EH + 4/3$ of $BH <$ arc BAE.

20. Similarly, the following rules for the mensuration of the hyperbola may be deduced.

Let ADB be a segment of a hyperbola [Barrow's figure is really half a segment], whose centre is C, axis AD, and base DB; and let $AD : CA = s : t$.

Then (1) $(2t + s)/(3t + 2s)$ of $AD . DB <$ segment ADB;
(2) $2/3$ of $AD . DB >$ segment ADB.

NOTE

The results of §§ 19, 20, for which Barrow omits any hint as to proof, are thus obtained.

§ 19 (1) A paraboliform whose exponent is $(t - s)/(2t - s)$ can be drawn, touching the circle BAE at B, A, and E, and lying completely outside it; the area of it cut off by the chord BE is, by § 11, equal to $(2t - 2s)/(3t - 2s)$ of $AD . BE$. (3) A parabola is a paraboliform whose exponent is $1/2$, and the area of the segment is $2/3$ of $AD . BE$. (2) and (4) follow from (1) and (3) by using obvious relations for the circle, and are not obtained independently. This explains why there are only two formulæ given for the hyperbola, and these are formulæ for the segment; for there are no corresponding simple relations for the hyperbola that connect the sector or segment with the arc.

§ 20. In a similar way, the two limits for the hyperbolic segment are obtained from a paraboliform whose exponent is $(t+s)/(2t+s)$, and a parabola.

The formulæ of (1) and (2) for the circle reduce to the trigonometrical equivalent $\alpha < \sin\alpha\,.\,(2+\cos\alpha)/(1+2\cos\alpha)$ in which the error is approximately $\alpha^5/45$; the formulæ of (3) and (4) reduce to the much less exact equivalent $\alpha > \sin\alpha\,.\,(2+\cos\alpha)/3$, where α is the half-angle. Thus Barrow's formula is a slightly more exact approximation than that of Snellius, namely, $3\sin 2\alpha/2(2+\cos 2\alpha)$, where the error is approximately $4\alpha^5/45$, and is in defect; Barrow, in § 29, obtains Snellius' formula in the more approximate form $3\sin\alpha/(2+\cos\alpha)$. Hence Barrow's formula and Snellius' formulæ give together good upper and lower limits to the value of the circular measure of an angle. The equivalent to the first formula for the hyperbola is $\sin^{-1}(\tan\alpha) > 3\tan\alpha/(1+2\cos\alpha)$; the error being again of the order α^5.

21. Further, let BAE be the segment of a circle whose centre is C, axis AD, centre of gravity K; also let AD : CA = $s : t$, and HD : AD $= 2t-s : 5t-2s$; then HD will be greater than KD.*

22. Let the point L be the centre of gravity of the parabola (such as was discussed in § 18); then L will be below K; *i.e.* KD is greater than two-fifths of AD.*

23. Let BAE be a segment of a hyperbola whose centre is C, axis AD, base BE, and centre of gravity K; also let AD : CA $= s : t$, and HD : AD $= 2t+s : 5t+2s$; then HD is less than KD.*

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* See note at foot of next page.

24. The centre of gravity of the parabola, L say, lies above K; *i.e.* KD is less than two-fifths of AD.*

25. Lest the present method of research, owing to the great number of methods of this kind for measuring the circle, may seem to be of little account, we will add one or two riders (if only for the sake of these, the few theorems given would deserve employment); from which indeed Maxima and Minima of things of a kind may be determined in a great number of cases.

Let ABZ be a semicircle whose centre is C; also let ADB be a segment; and to this let a paraboliform AFB be adscribed, whose exponent is n/m, where $\mathrm{AD} : \mathrm{CA} = m - 2n : m - n$.

If the parameter of the paraboliform † (that is a straight line such that some power of it multiplied by a power of the axis of the segment, AD say, produces a power of the ordinate, DB say) is p; then p will be a maximum of its kind.

For, if any straight line GE is drawn parallel to DB, and a paraboliform of like kind to AFB is supposed to be applied to GE, of which the parameter is called q; then, since the paraboliform AFB touches the circle externally, $\mathrm{GF} > \mathrm{GE}$. $\therefore\ \mathrm{GF}^m > \mathrm{GE}^m$, or $p^{m-n} . \mathrm{AG}^n > q^{m-n} . \mathrm{AG}^n$, $\therefore p > q$.

It should be observed that $p^{2(m-n)} = \mathrm{ZD}^m . \mathrm{AD}^{m-2n}$ and

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* These limits are not remarkable for close approximation unless the segment is very shallow. Thus if the arc is one-third of the circumference, the limits for the circle are only 2AD/5 and 3AD/5.

† It is to be observed that Barrow here indicates that the equation to the paraboliform is, in general $y = ax^{m/n}$.

$q^{2(m-n)} = \mathrm{ZG}^m \,.\, \mathrm{AG}^{m-2n}$, hence $\mathrm{ZD}^m \,.\, \mathrm{AD}^{m-2n} > \mathrm{ZG}^m \,.\, \mathrm{AG}^{m-2n}$; that is $\mathrm{ZD}^m \,.\, \mathrm{AD}^{m-2n}$ is a maximum.*

Example 1. Let $n = 1$, and $m = 3$; then $p^4 = \mathrm{ZD}^3 \,.\, \mathrm{AD} = \mathrm{ZD}^2 \,.\, \mathrm{BD}^2$, or $p^2 = \mathrm{ZD} \,.\, \mathrm{BD}$; and $\mathrm{AD} = \mathrm{CA}/2$.

Example 2. Let $n = 3$, and $m = 10$; then $p^{14} = \mathrm{ZD}^{10} \,.\, \mathrm{AD}^4$, or $p^7 = \mathrm{ZD}^5 \,.\, \mathrm{AD}^2 = \mathrm{ZD}^3 \,.\, \mathrm{BD}^4$, and $\mathrm{AD} = 4\mathrm{CA}/7$.

26. Again, let AEB be an equilateral hyperbola whose centre is C, and axis ZA; and to it let a paraboliform AFB, whose exponent is n/m and parameter p, be adscribed (with a base DB); also suppose that $\mathrm{AD} : \mathrm{CA} = 2n - m : m - n$; then p will be a minimum of its kind.

It is to be noted that $p^{2(m-n)} = \mathrm{ZD}^m \,.\, \mathrm{AD}^{2n-m}$, and also (as in § 25) $q^{2(m-n)} = \mathrm{ZG}^m \,.\, \mathrm{AG}^{2n-m}$; hence $\mathrm{ZD}^m \,.\, \mathrm{AD}^{2n-m}$ is a minimum.†

As in the preceding I have touched upon the mensuration of the circle, what if I add incidentally a few theorems bearing upon it, which I have by me? The following general theorem must, however, be given as a preliminary.

27. Let AGB be any curve whose axis is AD, and let the straight lines BD, GE be ordinates to it. Then the arc AB will bear a greater ratio to the arc AG than the straight line BD to the straight line GE.

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* This is equivalent to the algebraical theorem that, if $x + y =$ a constant then $x^r \,.\, y^s$ is a maximum when $x/r = y/s$.

† This is equivalent to the algebraical theorem that, if $x - y =$ a constant, then x^r/y^s is a minimum when $x/r = y/s$.

The proof of this theorem is generally ascribed to Ricci, who proved it algebraically in 1666, and used it to draw the tangent to the general paraboliform; thus we see that Barrow's proof was independent of Ricci, even if Barrow had not discovered it before Ricci; *cf.* "many years ago."

28. Let AMB be a circle, of which the radius is CA, and let DBE be a straight line perpendicular to CA; also let ANE be a curve such that, when any straight line PMN is drawn parallel to DE, cutting the circle in M and the curve in N, the straight line PN is equal to the arc AM. Then the parabola described with axis AD and base DE will fall altogether outside the curve ANE.

29. From the preceding, and from what is commonly known about the dimensions of the spaces ADB, ADE,* the following formula may be easily obtained:

$$3\mathrm{CA}\,.\,\mathrm{DB}/(2\mathrm{CA}+\mathrm{CD}) < \text{arc AB}.$$

Further, if the arc AB is supposed to be one of 30 degrees, and $2\mathrm{CA} = 113$, then the whole circumference, calculated by this formula, will prove to be greater than 355 less a fraction of unity.

30. Hence also, being given the arc AB, let arc $\mathrm{AB} = p$, $\mathrm{CA} = r$, and $\mathrm{DB} = e$; then the following equation may be used to find the right sine DB:

$$3r^2p^2/(9r^2+p^2) = 12r^2pe/(9r^2+p^2) - e^2;$$

or, substituting k for $3r^2p/(9r^2+p^2)$, we have

$$kp = 4ke - e^2; \quad \text{or} \quad 2k - \sqrt{(4k^2 - kp)} = e.$$

31. Let AMB be a circle whose radius is CA, and let the straight line DBE be perpendicular to CA; let also the curve ANE be a part of the cycloid pertaining to the circle AMB; and lastly let a parabola AOE be drawn with axis AD and base DE. Then the parabola will fall altogether within the cycloid.

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* See note at the end of this lecture.

32. From the preceding, and from what is generally known about the dimensions of circles and cycloids,* the following formula may be obtained;

$$(2CA \,.\, DB + CD \,.\, DB)/(CA + 2CD) > \text{arc } AB.$$

Further, if the arc AB is one of 30 degrees, and $2CA = 113$, it may be shown by this formula that the whole circumference is less than 355 plus a fraction.

You see then that, from the two formulæ stated, there results immediately the proportion of the diameter to the circumference as given by Metius.

33. Since in this straying from the track, the cycloid has brought itself under notice, I will add the following theorem;—I am not aware that it has been anywhere observed by those who have written so profusely on the cycloid.

If the space ADEG is completed (in § 31), the space AEG will be equal to the circular segment ADB.

The proof I shall leave out, nor shall I wander further from my subject.

34. Let two circles AIMG, AKNH touch one another at A, and have a common diameter AHG; and let any straight line DNM be drawn perpendicular to AHG. Then the segment AIMD will bear to the segment AKNH a less ratio than the straight line DM to the straight line DN.

35. Let YFZT be an ellipse, of which YZ and HT are the conjugate axes; and let the straight line DC be parallel to

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* See note at the end of this lecture.

the major axis YZ, and let the circle DFCV, whose centre is K a point on the minor axis FT, pass through the points D, F, C; then I say that the part DOFPC of the circle will lie within the part DMFNC of the ellipse.

36. Let DEC be a segment of a circle whose centre is L; and, any point F being taken in its axis GE, let DMFC be a curve such that, when any straight line RMS is drawn parallel to GE, RS : RM = GE : GF; then DMFC is an ellipse thus determined:—Find H, such that EG : FG = GL : GH; through H draw YHZ parallel to DC, and let HY equal LE; then HY, HF are the semi-axes of the ellipse.

This is held to have been proved by Gregory St Vincent, Book IV, Prop. 154.

Corollary.—Hence, segment DEC : segment DMFC = EG : FG.

37. Let DEC, DOFC be portions of two circles having a common chord DC and axis GFE; then the greater portion DEC will bear to the portion DOFC a greater ratio than that which the axis GE bears to the axis GF.

Note

In § 29, Barrow gives no indication of the source of the "known dimensions," and there is also probably a misprint; for the "spaces ADB, ADE," we should read the "spaces ANED, AOED," unless Barrow intended ADE to stand for both the latter spaces. If so, we have from § 2, area AOED = 2/3 of AD . DE, and the area of ANED can thus be found by Barrow's methods:—

Complete the rectangle EDCF, and draw QRS parallel and indefinitely near to PMN; draw SVZ parallel to AC, cutting

PN, CF in V, Z, and RT parallel to AC, cutting PN, CF in T, Y; then we have CP : CM = MT : MR = MT : NV,

$\therefore$ CP . NV = CM . MT.

Hence area ANED + CD . DE
= the sum of VN . CP + . . .
= the sum of CM . MT + . . .
= CM . (the sum of MT + . . .)
= CM . BD;
Area ANED = CA . BD − CD . DE.
$\therefore$ CA . BD − CD . DE < area AOED
< 2/3 of AD . DE,
$\therefore$ 3CA . BD < (3CD + 2AD) . DE
or (2CA + CD) . arc AB.

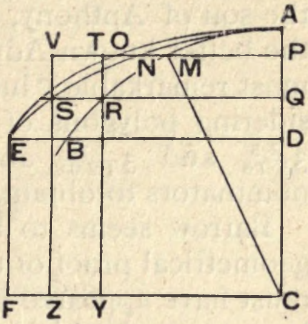

Fig. 151.

It should be observed that the equivalent of the expression for the area ANED is

$$\int a\, cos^{-1}\, x/a\, dx = x\, cos^{-1}\, x/a - a \,.\, \sqrt{}(a^2 - x^2).$$

The formula finally obtained by Barrow, if we put 2ϕ for the angle subtended at the centre of the circle by the arc AB, reduces to $2\phi > 3\, sin\, 2\phi/(2 + cos\, 2\phi)$, which is the formula of Snellius; this, as I have already noted, has an error of the order ϕ^5; a handier result is obtained by taking $a = 2\phi$, when it becomes $a > 3 \sin a/(2 + \cos a)$.

For § 32, since MN (of this theorem) = arc AM = PN (in fig. 151); hence

area of cycloid = area of AMBD + area of ANED (fig. 151)
= (CA . arc AB − CD . BD)/2 + CA . BD − CD . arc AB.

The remark made by Barrow in § 33 indicates with almost certainty that the above was his method for the cycloid.

Now, since the area of the cycloid is less than the area of the corresponding parabola, which is 2/3 of AD . DE or 2/3 of AD . (DB + arc AB); hence we obtain

arc AB < (2CA . DB + CD . DB)/(CA + 2CD).

This is equivalent to $a < sin\, a(2 + cos\, a)/(1 + 2\, cos\, a)$, a limit obtained before in § 19. Thus Barrow has here two very close limits, one in excess and the other in defect, each having an error of the order a^5.

The results obtained, by the use of these approximate formulæ, with the convenient angle of 30 degrees are in fact 355·6 and 354·8. The formula obtained by "Adrian, the son of Anthony, a native of Metz (1527), and father of the better known Adrian Metius of Alkmaar" is one of the most remarkable "lucky shots" in mathematics. By considering polygons of 96 sides, Metius obtained the limits $3\frac{15}{106}$ and $3\frac{17}{120}$, and then added numerators and denominators to obtain his result $3\frac{32}{226}$ or $3\frac{16}{113}$! ! !

Barrow seems to be content, as usual, with giving the geometrical proof of the formula obtained by Metius; which must have appeared atrocious to him as regards the method by means of which the final result was obtained from the two limits. [If only Barrow had not had such a distaste for long calculations,] such as that by which Briggs found the logarithm of 2 (he extracted the square root of 1·024 forty-seven times successively and worked with over thirty-five places of decimals), [it would seem to be impossible that Barrow should not have had his name mentioned with that of Vieta and Van Ceulen and others as one of the great computers of π.] For he here gives both an upper and a lower limit, and therefore he is only barred by the size of the angle for which he can determine the chord. Now, he would certainly know the work of Vieta; and this would suggest to him that a suitable angle for his formulæ would be $\pi/2^n$, where n was taken sufficiently large. For Vieta's work would at once lead him to the formulæ

$$2 \cos \pi/2^n = \sqrt{}[2+\sqrt{}\{2+\sqrt{}(2+\ .\ .\ .\ .\ .\ .\ .)\}],$$
$$2 \sin \pi/2^n = \sqrt{}[2-\sqrt{}\{2+\sqrt{}(2+\ .\ .\ .\ .\ .\ .\ .)\}],$$

where there are $n-1$ root extractions in each case.

If, then, he took n to be 48, his angle would be less than $1/2^{46}$, and the error in his values would be less than $1/2^{234}$; this is about 10^{-45}; hence Barrow has practically in his hands the calculation of π to as many decimal places as the number of square root extractions he has the patience to perform and the number of decimal places that he is willing to use.]

LECTURE XII

General theorems on Rectification.

General Foreword.—We will now proceed with the matter in hand; and in order that we may as far as possible save time and words, it is to be observed everywhere in what now follows that AB is some curved line, such as we shall draw, of which the axis is AD; to this axis all the straight lines BD, CA, MF, NG are applied perpendicular; the arc MN is indefinitely small; the straight line $\alpha\beta$ = arc AB, the straight line $\alpha\mu$ = arc AM, and $\mu\nu$ = arc MN; also lines applied to $\alpha\beta$ are perpendicular to it. On this understanding,

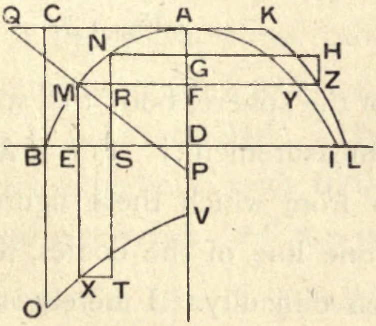

Fig. 156.

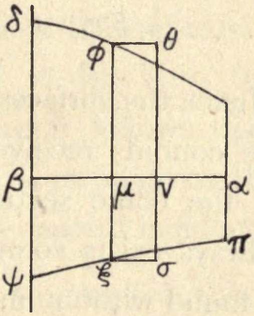

Fig. 157.

1. Let MP be perpendicular to the curve AB, and the

lines **KZL**, $\alpha\phi\delta$ such that **FZ** = **MP** and $\mu\phi$ = **MF**. Then the spaces $\alpha\beta\delta$, **ADLK** are equal.*

2. Hence, if the curve **AMB** is rotated about the axis **AD**, the ratio of the surface produced to the space **ADLK** is that of a circumference of a circle to its diameter; whence, if the space **ADLK** is known, the said surface is known.

Some time ago we assigned the reason why this was so.

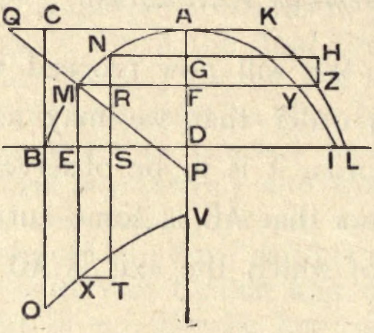

Fig. 156.

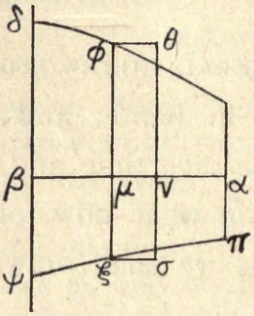

Fig. 157.

3. Hence the surfaces of the sphere, both the spheroids, and the conoids receive measurement.† For, if **AD** is the axis of the conic section from which these figures arise, there always exists some one line of the conics, **KZL**, that can be found without much difficulty. I merely state this, for it is now considered as common knowledge.

4. With the same hypothesis, let **AYI** be a curve such that the ordinate **FY** is a mean proportional between the corresponding **FM**, **FZ**. Then the solid formed by the rotation of the space $\alpha\delta\beta$ about the axis $\alpha\beta$ will be equal

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* The equivalent is $yds = y \,.\, (ds/dx)dx$.

† For the circle, the figure **ADLK** is a rectangle; and the area of a zone is immediately deducible; and so on.

to the solid formed by the rotation of the space ADI about the axis AD.

5. By similar reasoning, it may be deduced that, if FY is supposed to be a bimedian between FM and FZ, the sum of the cubes of the applied lines, such as $\mu\phi$, from the curve $\alpha\phi\delta$, to the straight line $\alpha\beta$ is equal to the sum of the cubes on the lines applied to the straight line AD from the curve AYI. Similarly, the theorem holds for other powers.

6. Further, with the same hypothesis, let the curve VXO be such that EX = MP; and let the curve $\pi\xi\psi$ be such that $\mu\xi$ = PF. Then the space $\alpha\pi\psi\beta$ = the space DVOB.

7. Observe also that, if the curve AB is a parabola, whose axis is AD and parameter R; then the curve VXO will be a hyperbola, whose centre is D, semi-axis DV, and the parameter of this axis equal to R. Also the space $\alpha\beta\psi\pi$ will be a rectangle. Hence it follows that, being given the hyperbolic space DVOB, is to be given the curve AMB, and vice-versa. All this is remarked incidentally.*

8. It should also be possible to observe that all the squares on the lines applied to the straight line $\alpha\beta$, taken together, from the curve $\pi\xi\psi$, are equal to all the rectangles such as PF . EX, applied to the line DB (or calculated); the cubes on $\mu\xi$ are equal to the sum of PF^2 . EX, etc.; and so on.

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* Yet it has an important significance; for it is the first indication that Barrow is seeking the connection between the problem of the rectification of the parabola and that of the quadrature of the hyperbola. He is not quite satisfied with this result, but finally succeeds in § 20, Ex. 3.

9. Also it may be noted that, PMQ being produced, if FZ is supposed to be equal to PQ, and $\mu\phi$ to AQ; then the space $\alpha\beta\delta$ is equal to the space ADLK.

10. Further, let the straight line MT touch the curve AB, and let the curves DXO, $\alpha\phi\delta$ be such that EX = MT and $\mu\phi$ = MF. Then the space $\alpha\beta\delta$ is equal to the space DXOB.

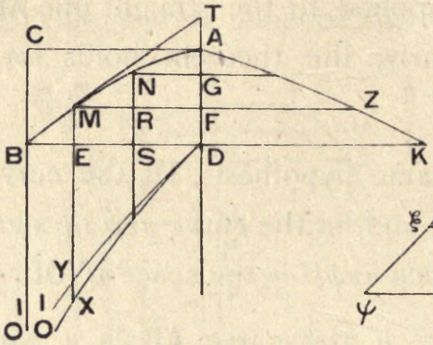


Fig. 158.

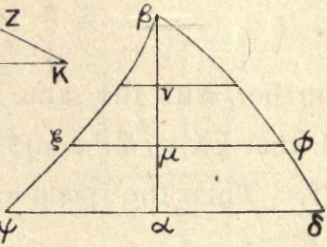


Fig. 159.

11. Hence again, the surface of the solid formed by the rotation of the space ABD about the axis AD bears to the space ADOB the ratio of the circumference of a circle to its radius; therefore, if one is known, the other becomes known at the same time.

Hence again one may measure the surfaces of spheroids and conoids.

12. If the line DYI is such that $EY^2 = EX \cdot MP$; then the solid formed by the rotation of the space $\alpha\beta\delta$ about the axis $\alpha\beta$ is equal to the solid formed by the rotation of the space DBI about the axis IB.

13. By similar reasoning, one may compare the sums of the cubes and other powers of the ordinates with spaces computed to the straight line DB.

14. Moreover, let the lines AZK, $\alpha\xi\psi$ be such that FZ = MT and $\mu\xi$ = TF; then the space $\alpha\beta\psi$ will be equal to ADK.

15. Also the sum of the squares on the applied lines $\mu\xi$ will be equal to the sum of the rectangles TF . FZ; the sum of the cubes on $\mu\xi$ to the sum of TQ . FZ, . . . (considering them to be computed to the straight line AD); and so on for the other powers.

16. Again let the straight line QMP be perpendicular to the curve AMB; and let $\beta\delta$ = BD; complete the rectangle $\alpha\beta\delta\zeta$; then let the curve KZL be such that FZ = QP. Then the rectangle $\alpha\beta\delta\zeta$ is equal to the space ADLK.

Therefore, if the space AKLD is known, the quantity of the curve AMB is also known.

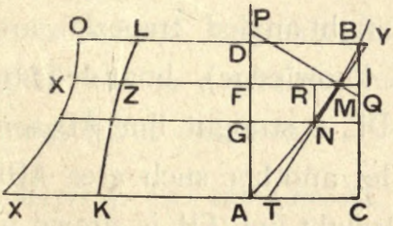


Fig. 160.

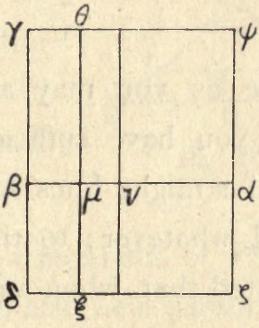


Fig. 161.

17. Also, let the straight line TMY be supposed to touch the curve AMB, and let $\beta\gamma$ be made equal to BC, and the rectangle $\alpha\beta\gamma\psi$ be completed; let then the curve OXX be such that FX = TY. Then the space ADOXX . . .—indefinitely continued—will be equal to the rectangle $\alpha\beta\gamma\psi$.

Hence, again, if the space ADOXX . . . has been ascertained, then the curve AMB becomes known.

18. Moreover, if any determinate length R is taken, and $\beta\delta$ is taken equal to R; and if the curve OXX is such that MF : MP = R to FX; then the rectangle $\alpha\beta\delta\zeta$ will be equal to the space ADOXX. Also, if this space is found, the curve is forthwith known.

Many other theorems like this could be set down; but I fear that these may already appear more than sufficient.

19. It should be observed, however, that all these theorems are equally true, and can be proved in exactly the same way if the curve AMB is convex to the straight line AD.

20. Also, from what has been shown, an easy method of drawing curves (theoretically) is obtained, such as admit of measurement of some sort; in fact, you may proceed thus:—

Take as you may any right-angled trapezial area (of which you have sufficient knowledge), bounded by two parallel straight lines AK, DL, a straight line AD, and any line KL whatever; to this let another such area ADEC be so related that, when any straight line FH is drawn parallel to DL, cutting the lines AD, CE, KL in the points F, G, H, and some determinate straight line Z is taken, then the

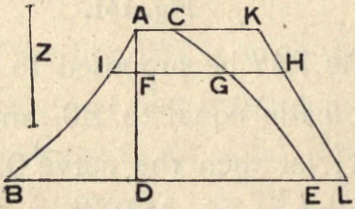


Fig. 162.

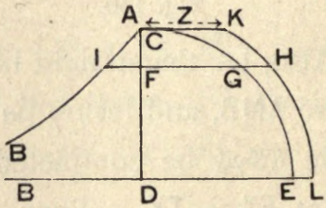


Fig. 163.

square on FH is equal to the squares on FG and Z. Moreover, let the curve AIB be such that, if the straight line GFI is produced to meet it, the rectangle contained by Z and FI is equal to the space AFGC; then the rectangle contained by Z and the curve AB is equal to the space ADLK. The method is just the same, even if the straight line AK is supposed to be infinite.

Example 1. Let KL be a straight line, then the curve CGE is a hyperbola. (Fig. 162.)

Example 2. Let the line KL be the arc of a circle whose centre is D, and let AK = Z; then the curve AGE will be a circle; and the arc $AB = AD/2 + (DL/2AK) \cdot \text{arc } KL$. (Fig. 163.)

Example 3. Let the line KL be an equilateral hyperbola, of which the centre is A, and the axis AK = Z; then CGE will be a straight line, and the curve AB a parabola.

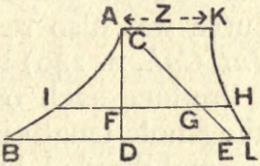


Fig. 164.

Example 4. Let the line KL be a parabola, of which the axis is AD; then the line CGE will also be a parabola, and the curve AB one of the paraboliforms.

Example 5. Let the curve KL be an inverse or infinite paraboliform (for instance, such that $FH^2 = Z^3/AF$); then the curve AB will be a cycloid, pertaining to the circle whose diameter is equal to Z. (See figure on page 164.)

Perhaps, if you consider, you may think of some examples that are neater than these.

NOTE

The chief interest in the foregoing theorems lies in the last of all. The others are mainly theorems on the change of the variable in integration (or rather that the equality $(\delta z/\delta x) . \delta y = (\delta y/\delta x) . \delta z$ holds true in the limiting form for the purposes of integration, although of course Barrow does not use Leibniz' symbols); and secondly, the application of this principle to obtain *general* theorems on the rectification of curves, by a conversion to a quadrature. It must be borne in mind that Barrow's sole aim, *expressly stated*, was to obtain *general* theorems; and that he merely introduced the cases of the well-known curves as examples of his theorems; and to obtain the gradient of the tangent of a curve in general is the foundation of the differential calculus.

In 1659, Wallis showed that certain curves were capable of "rectification"; the first-fruits of this was the rectification of the semi-cubical parabola by William Neil in 1660, by the use of Wallis' method. Almost simultaneously this curve was also rectified by Van Huraet (see Williamson's *Int. Cal.*, p. 249) by the use of the geometrical theorem:—"Produce each ordinate of the curve to be rectified until the whole length is in a constant ratio to the corresponding normal divided by the old ordinate, then the locus of the extremity of the ordinate so produced is a curve whose area is in a constant ratio to the length of the given curve."

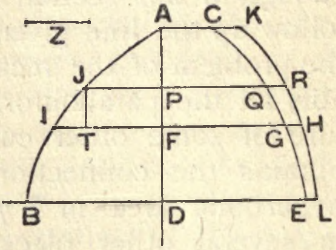
Now this theorem is identical with the theorem of § 18; hence, remembering that the semi-cubical parabola, whose equation is $\mathrm{R} . y^2 = x^3$, is one of those paraboliforms of which Barrow is so fond, and for which, as we have seen, he could find both the tangent at any point and the area under the curve between any two ordinates, noting also the examples given to the theorem of § 20, it is beyond all doubt that Barrow must have perceived that for this particular paraboliform his curve OXX . . . (fig. 160) was the parabola $4y^2 = \mathrm{R} . (9x + 4\mathrm{R})$. Why then did not Barrow give the result? The answer, I think, is given in his own remark before Problem IX in Appendix III, "*I do not like to put my sickle into another man's harvest*," where he refers to the work of James Gregory on involute and evolute figures.

Moreover, this supposition may set a date to this section, namely not before 1659, and not very much later than 1661. For from his opening remarks to the Appendix to Lect. XI, we can gather that it was Barrow's habit to read the work of his contemporaries as soon as he could get them, and try to "go one better," and there are indications enough in this section to show that Barrow was trying to follow up the line given by § 7, to obtain the reduction of the problem of the rectification of the parabola (and probably all the paraboliforms in general as well) to a quadrature of some other curve; we see, for instance, that he obtains the connection between a parabolic arc and a hyperbolic area in § 7, and this connection is obtained in several other places by different methods. He also seeks *general* theorems in which the quadrature belongs to one of the paraboliforms or the hyperboliforms (curves that can be obtained from a rectangular hyperbola in the same way as the paraboliforms are obtained from a straight line in Lect. IX, § 4); and the result of using these curves, whose general equation is $y^m x^n = \mathsf{R}^{m+n}$, is seen in § 20, Ex. 5, where he takes $m = 2$, and $n = 1$, and the derived curve is the Cycloid. He does not state that thus he has rectified the cycloid, apparently because in Prob. 1, Ex. 2 of App. III, he has obtained it in a much simpler manner as a special case of another *general theorem.* (See critical note that follows this problem.)

The great interest, however, of this section centres in the question of the manner in which Barrow obtained the construction for § 20. There is nothing leading to it in any theorem that has gone before it in the section; the only case in which he has used the construction of a subsidiary curve, such that the difference of the squares on the ordinates of the two curves is constant, is in Lect. VI, §§ 22, 23, and then his original curve is a straight line. The only conclusion that I can come to is that he uses his *general theorem on rectification* (Lect. X, § 5) *analytically* thus:—

If $\mathsf{Z} \,.\, (d\mathsf{S}/dx) = y$, where S is the arc of the curve to be rectified, and Y its ordinate, we must have $\mathsf{Z} \,.\, (d\mathsf{Y}/dx)$ equal to $\sqrt{(y^2 - \mathsf{Z}^2)}$, and therefore $\mathsf{Z} \,.\, \mathsf{Y} = \int\sqrt{(y^2 - \mathsf{Z}^2)}dx$. The given construction is an immediate consequence.

Of course Barrow knew nothing about the notation dY/dx or $\int\sqrt{(y^2 - Z^2)}dx$; his work would have dealt with small finite arcs and lines; but the pervading idea is better represented for argument's sake by the use of Leibniz' notation. I suggest that Barrow's proof would have run in something like the following form :—



Draw JPQR parallel and very near to IFGH, cutting the curves as shown in the adjoining diagram, and draw JT perpendicular to IH; then

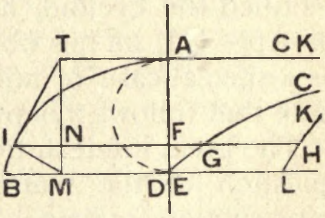
$$Z\,.\,IT = \text{area } PFGQ = PF\,.\,FG;$$

$$\therefore\ Z^2\,.\,IJ^2 = Z^2\,.\,IT^2 + Z^2\,.\,TJ^2 = PF^2\,.\,FG^2 + Z^2\,.\,PF^2;$$

$$\therefore\ Z\,.\,IJ = PF\,.\,FH.$$

Hence, summing, $Z\,.\,\text{arc } AB = \text{area } ADLK$.

That Barrow had, in § 20, Ex. 5, really rectified the cycloid is easily seen from the adjoining diagram. Barrow starts, I suppose, with the property of the cycloid that, if IT, IM are the tangent and normal at I, then TM is perpendicular to BD. Let $AD = Z$, then since $Z\,.\,IT/TN = FH$,* we have $FH^2 = Z^2.\,IT^2/TN^2 = Z^2\,.\,TM\,.\,TN/TN^2 = Z^2\,.\,TM/TN = Z^3/AF$.



The area under the curve KHL is given as proportional to the ordinate of what I may call its integral curve (see note to Lect. XI, § 2), and is easily shown to be $2AF\,.\,FH$.

Hence $\text{arc } AI = \text{area } AFHK/Z = 2AF\,.\,FH/Z = 2IT$; that is, equal to twice the chord of the circle parallel to TI, which is also equal to it.

---

* This follows at once from the figure at the top of the page; for, $Z\,.\,IJ = PF\,.\,FH$, and $IT : TN$ (in the lower figure) is equal to $IJ : JT$ (in the upper figure); and this is equal to $IJ : PF$ or $FH : Z$; hence, in lower figure $IT : TN = FH : Z$.

APPENDIX I

Standard forms for integration of circular functions by reduction to the quadrature of a hyperbola.

Here, although *it is beyond the original intention to touch on particular theorems in this work*;* and indeed to build up these general theorems with such corollaries would tend to swell the volume beyond measure; yet, *to please a friend who thinks them worth the trouble*,* I add a few observations on tangents and secants of a circle, most of which follow from what has already been set forth.

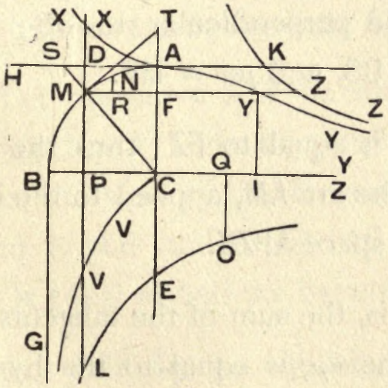


Fig. 166.

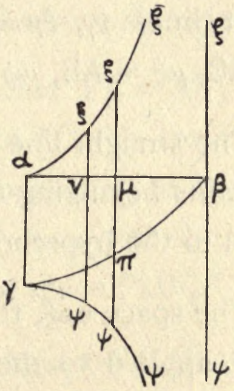


Fig. 167.

General Foreword. — Let ACB be a quadrant of a circle, and let AH, BG be tangents to it; in HA, AC produced take AK, CE each equal to the radius CA; let the hyperbola KZZ be described through K, with asymptotes AC, CZ; and let the hyberbola LEO be described through E, with asymptotes BC, BG.

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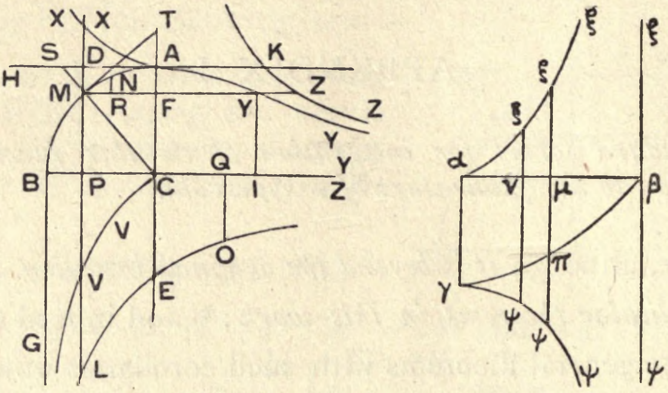
* Observe the words of the opening paragraph which I have italicized.



Also let an arbitrary point M be taken in the arc AB, and through it draw CMS cutting the tangent AH in S, MT touching the circle, MFZ parallel to BC, and MPL parallel to AC. Lastly, let $\alpha\beta$ = arc AB, $\alpha\mu$ = arc AM; let the straight lines $\alpha\gamma$, $\xi\mu\pi\psi$ be perpendicular to $\alpha\beta$; and let $\alpha\gamma$ = AC, $\mu\xi$ = AB, $\mu\psi$ = CS, and $\mu\pi$ = MP.

1. The straight line CS is equal to FZ; thus the sum of the secants belonging to the arc AM, applied to the line AC, s equal to the hyperbolic space AFZK.

2. The space $\alpha\mu\xi$, that is, the sum of the tangents to the arc AM, applied to the line $\alpha\mu$, is equal to the hyperbolic space AFZK.

3. Let the curve AXX be such that PX is equal to the secant CS or CT; then the space ACPX, that is the sum of the secants belonging to the arc AM, applied to the line CB, is twice the sector ACM.

4. Let CVV be a curve such that PV is equal to the tangent AS; then the space CVP, that is, the sum of the tangents belonging to the arc AM, applied to the straight line CB, is equal to half the square on the chord AM.*

5. Let CQ be taken equal to CP, and QO be drawn parallel to CE, meeting the hyperbola LEO in O; then the hyperbolic space PLOQ multiplied by the radius CB (or the cylinder on the base PLOQ of height CB) is double the sum of the squares on the straight lines CS or PX, belonging to the arc AM, and applied to the straight line CB.*

6. Hence the space $\alpha\gamma\psi\mu$, that is, the sum of the secants of the arc AM applied to the line $\alpha\beta$, is equal to half the hyperbolic space PLOQ.*

7. All the squares on the straight lines $\mu\psi$, applied to $\alpha\mu$, are equal to CA . CP . PX, that is, equal to the parallelepiped on the rectangular base APCD whose altitude is CS.

8. Let the curve AYY be such that FY = AS; then, if a straight line YI is drawn parallel to AC, the space ACIYYA (that is, the sum of the tangents belonging to the arc AM, applied to the straight line AC, together with the rectangle FCIY) is equal to half the hyperbolic space PLOQ.*

9. Let ERK be an equilateral hyperbola (that is, one having equal axes), and let the axes be CED, CI; also let KI, KD be ordinates to these; let EVY be a curve such that,

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* These theorems are not at first sight of any great interest; they appear only to be a record of Barrow's attempts to connect the quadrature of the hyperbola in some way with the circle. But later, when we find that Barrow has the area under the hyperbola, their importance becomes obvious. (See critical note following App. III, Probs. 3, 4.)

when any point R is taken at random on the hyperbola, and a straight line RVS is drawn parallel to DC, then SR, CE, SV are in continued proportion; join CK; then the space CEYI will be double the hyperbolic sector DCE.

10. Returning now to the circular quadrant ACB, let CE = CA; and with axis AE, and parameter also equal to AE, let the hyperbola EKK be described; now let the curve AYY be supposed to be such that, when any ordinate MFY is drawn, FY is equal to the tangent AS; draw YIK, cutting CZ in I and the hyperbola in K, and join CK, then the space ACIYA is double the hyperbolic sector ECK.

11. COROLLARY.—Hence, if with pole E, a chord CB, and a sagitta CA, a conchoid AVV is described; and if YFM produced meets it in V; then MV = FY; and thus the space AMV is equal to the space AFY.

12. Whence the dimensions of conchoidal spaces of this kind become known.

13. Let AE be a straight line perpendicular to RS (cutting it in C); and let CE = CA; let AZZ, EYY be two conchoids, conjugate to one another, described with the same pole E and a common chord RS; from E draw any straight line EYZ, cutting EYY, AZZ, RS in the points Y, Z, I; also let EKK be an equilateral hyperbola, with centre C and semi-axis CE; draw IK parallel to AE and join CK.

Then the four-sided space, bounded by AE, YZ, and the conchoidal arcs EY, AZ is equal to four times the hyperbolic sector ECK.

14. We will also add to these the following well-known measurement of cissoidal space.

Let AMB be a semicircle whose centre is C, and let the straight line AH touch it; and let AZZ be the cissoid that is congruent to it, having this property, that, if any point M is taken in the circumference AMB, and through it the straight line BMS is drawn (cutting AH in S), and also a straight line MFZ, cutting the cissoid AZZ in Z, MZ = AS; then in a straight line $\alpha\beta$ take $\alpha\mu$ equal to the arc AM, and to $\alpha\mu$ let straight lines $\mu\xi$ be applied perpendicular, and equal to the versines AF of the arc AM. Then the trilinear space MAZ is double the space $\alpha\mu\xi$. Hence, since the dimensions of the space $\alpha\mu\xi$ are generally known, and indeed can be easily deduced from the preceding theorems, therefore the dimension of the cissoidal space MAZ is obtained. Anyone may make the calculation who wishes to do so.

The following rider will close this appendix.

15. Let ACB be a quadrant of a circle, and let AH, BG touch the circle; also let the curves KZZ, LEO be hyperbolas, the same as those that have been used above; let the arc AM be taken, and let it be supposed to be divided into parts at an infinite number of points N; through these draw radii CN, and let the straight lines NX (drawn parallel to AH) meet them in the points X. Then the sum of the straight lines NX (taken along the radii) will equal to the space AFZK/(radius), and the sum of the straight lines NX (taken along parallels to AH) will be equal to the space PLQO/(3 . radius).

APPENDIX II

Method of Exhaustions. Measurement of conical surfaces.

For the sake of brevity combined with clearness, and especially for the latter, the proofs of the preceding theorems have been given by the direct method; by which not only is the truth firmly established, but also its origin appears more clearly. But for fear anyone, less accustomed to arguments of this nature, should hesitate to use them, we will add a few examples by which such arguments may be made sure, and by the help of which indirect proofs of the propositions may be worked out.

1. Let the ratios A to X, B to Y, C to Z, be any ratios, each greater than some given ratio R to S; then will the ratio of all the antecedents taken together to all the consequents taken together be greater than the ratio R to S.

2. Hence it is evident that, if any number of ratios are each of them greater than any ratio that can be assigned, then the sum of the antecedents bears a greater ratio to the sum of all the consequents than any ratio that can be assigned.

3. Let ADB be any curve, of which the axis is AD, and to this the straight line BD is applied; also let the straight line BT touch the curve, and let BP be an indefinitely small part of the line BD; draw PO parallel to DT, cutting the curve in N. Then I say that PN will bear to NO a ratio greater than any assignable ratio, R to S, say.

4. Hence, if the base BD is divided into an infinite number of equal parts at the points Z, and through these points are drawn straight lines parallel to DA, cutting the curve in E, F, G; and through the latter are drawn the tangents BQ, ER, FS, GT, meeting the parallels ZE, ZF, ZG, DA in the points Q, R, S, T; then the straight line AD will bear to all the intercepts EQ, FR, GS, AT taken together a ratio greater than any assignable ratio.

5. Among the results of this we have:—

All the lines EQ, FR, GS, AT taken together are equal to zero.

The lines ZE, ZQ; ZF, ZR; etc., are equal to one another respectively.

Also the small parts of the tangents BQ, ER, etc., are equal to the corresponding small parts of the curve, BE, EF, etc.; and they can be considered as coincident with one another.

Moreover, one may safely assume anything which evidently is consistent with these.

6. Again, let AB be any curve, of which the axis is AD, and let DB be applied to it; also let DB be divided into an indefinite number of equal parts at the points Z;

through these points draw straight lines parallel to AD, cutting the curve in the points X, and let these be met by straight lines ME, NF, OG, PH, drawn through the points X parallel to BD; also let the figure ADBMXNXOXPXRA, circumscribed to the segment ADB (contained by the straight lines AD, DB and the curve AB), be greater than any space S; then I say that the segment ADB is not less than the space S.

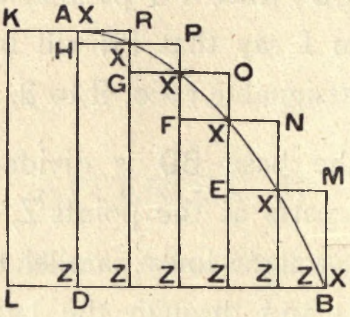

Fig. 176.

7. Also if it is supposed that the inscribed figure HXGXFXEXZDH is less than any space S; then I say that the segment ADB is not greater than S.

8. Hence, if there is any space, S say, the figure circumscribed to which is equal to the figure ADBMNOPRA, and also the figure inscribed to it is equal to the figure HGFEZDH; then the space S will be equal to the segment ADB. For, as has just been shown, it cannot be greater than it, nor can it be less.

Also these things can be altered to suit other modes of circumscription and inscription; it should be sufficient to have just made mention of this.

Note.—In § 6, Barrow uses the usual present-day method of translating the error for each rectangle across the diagram to sum them up on the last rectangle; another point of interest is the striking similarity between the figure used by Barrow and the figure used by Newton in Lemma II of Book I of the *Principia*, especially as Newton uses the four-part division of his base, which is usual with Barrow, whereas in this place Barrow, strangely for him, uses a five-part division of the base.

Method of Measuring the Surface of Cones

Let AMB be any curve, whose axis is AD, and C a given point in it, BD a straight line at right angles to it. Any point M in the curve being taken, draw ME touching the curve, and from C draw CG perpendicular to ME; also let CV be a straight line of given length, perpendicular to the plane ADB; join VG. Then VG will be perpendicular to MG. Also let RS be a line such that, if a straight line MIX is drawn parallel to AD, cutting the ordinate BD in I, and the line RS in X, then MP : ME = VG : IX; or, if the line AL is such that, when MPY is drawn parallel to BD, cutting the axis AD in P, and the line AL in Y, then PE : ME = VG : PY; then will either of the spaces BRSD or ADL be double the surface of the cone formed by straight lines through V that move along the curve AMB.

Example.—Let the curve AMB be an equilateral hyperbola, of which the centre is C, and let CV = CA = r, and CP = x (for it helps matters in most cases to use a calculation of this kind); join MC; then the rectangle BRSD is double the area AMBV of the cone.

This elegant example was furnished by that most excellent man, of outstanding ability and knowledge, Sir Francis Jessop, Kt., an Honorable ornament of our college, of which he was once a Fellow-commoner; I shall venture to adorn my book, as with a *jewel*, not indeed at his request, nor yet I hope against his wish, by means of his cleverly written work on this matter, kindly communicated to me.

PROPOSITION 1

If from a point E in the axis Am of a right cone ABCp, a straight line of unlimited length, EC, passes through the surface of the cone, and if with the end E kept at rest, the line EC is carried round until it returns to the place from which it started, so that always some part of it cuts the surface of the cone (say, through the hyperbola CFD and the straight lines DA, AC situated in the surface of the cone), the solid contained by the surface or surfaces generated by the straight line EC so moved and by the portion of the surface of the cone bounded by the line or lines CFD, DA, AC, which the straight line EC describes in the surface as it is carried round, will be equal to the pyramid of which the altitude is equal to En, the perpendicular drawn from the point E to the side of the cone, and base equal to that part of the conical surface bounded by the line or lines CFD, AD, AC, generated by the motion of the line EC.

PROPOSITION 2

Let ABCp be a right cone; let it be cut by the plane CFD parallel to its axis Am; let the straight lines AC, AD be drawn from the vertex of the cone to the hyperbolic line CFD; and upon the triangle ACD let the pyramid EACD be erected, having its vertex E in the axis of the cone; and let Eδ be perpendicular to the plane ACD and En to the side of the cone. Then I say that the conical surface bounded by the hyperbolic line CFD and the straight lines DA, AC is to the pyramid EACD on the base ACD as the altitude of the pyramid Eδ is to the perpendicular En.

PROPOSITION 3

Let ABCp be a given right cone; let it be cut by a plane (say, in the triangle qrt) and let this plane cut the axis of the cone produced beyond the vertex in the point q; also let the common intersection of it and the surface of the cone be the hyperbolic line rSt, and let straight lines Ar, At be drawn from A the vertex of the cone, from the point q a perpendicular qX to the side Ap of the cone produced, and from the point A a perpendicular AZ to the plane qrt. Then I say that the conical surface, bounded by the hyperbolic line rst and the straight lines rA, tA, is to the hollow hyperbolic figure qrStq as the perpendicular AZ is to the perpendicular qX.

PROPOSITION 4

Let ABhg be a given right cone; and let it be cut by a plane HFEG passing through the axis below the vertex; from the point H, where the plane cuts the axis of the cone, let HK be drawn perpendicular to any side of the cone, and from the vertex A a perpendicular AL to the plane HFEG. Then I say that the conical surface, bounded by the lines FEG, GA, AF is to the plane HGEF as the perpendicular AL is to the perpendicular HK.

APPENDIX III

Quadrature of the hyperbola. Differentiation and Integration of a logarithm and exponential. Further standard forms.

On looking over the preceding, there seems to me to be some things left out which it might be useful to add. Anyone can easily deduce the proofs from has already been given, and will obtain more profit from them thereby.

PROBLEM I

Let KEG be any curve of which the axis is AD, and let A be a given point in AD; find a curve, LMB say, such that, when any straight line PEM perpendicular to the axis AD cuts the curve KEG in E and the curve LMB in M, and AE is joined, and TM is a tangent to the curve LMB, then TM shall be parallel to AE.

The construction is made as follows:—Through any point R, taken in the axis AD, draw a straight line RZ perpendicular to AD; let EA produced meet it in S, and in the straight line EP take PY equal to RS; in this way the nature of the curve OYY is determined; then let the rectangle contained by AR and PM be equal to the space AYYP (or PM is equal to the space AYYP/AR). Then the curve AYYP shall have the proposed property.

It should also be easily seen that, other things remaining the same, if the curve QXX is such that, if EP cuts it in X, PX = AS; then the space AXXP is equal to the rectangle contained by AR and the arc LM, or space AXXP/AR = arc LM.

Example 1.—Let ADG be a quadrant of a circle; if EP is any straight line perpendicular to AD, join DE. It is required to draw the curve AMB such that, if EPM produced meets it in M, and MT touches the curve, then MT shall be parallel to DE.

The construction is as follows:—Draw AZ parallel to DG, and let DE produced meet it in S; let the curve AYY be such that, if PE produced meets it in Y, PY = AS; then take PM = space AYP/AD; and the construction is effected.

NOTE.—If the curve QXX is such that PX = DS (or if AQ = AD and QXX is a hyperbola bounded by the angle ADG), then arc AM . AD = space AQXP.

Example 2.—Let AEG be any curve whose axis is AD such that, when through any point E taken in it a straight line EP is drawn perpendicular to AD, and AE is joined, then AE is a given mean proportional between AR and AP of the order whose exponent is n/m. It is required to find the curve AMB, of which the tangent TM is parallel to AE.

Observe about the curve AM that $n : m$ = AE : arc AM.

Now, if $n/m = 1/2$ (or AE is the simple geometrical mean between AR and AP), then AEG will be a circle, and AMB the ordinary cycloid. *Hence the measurement of the latter comes out from a general rule.*

These also follow from the more general theorem added below.

NOTE

At first sight the foregoing proposition, stated in the form of a problem, but (by implication in the note above) referred to by Barrow as a theorem, would appear to be an attempt at an inverse-tangent problem. But "the sting is in the tail"; this, and most of those which follow, are really further attempts to rectify the parabola and other curves, by obtaining a quadrature for the hyperbola. That this is so is fairly evident from the note to Ex. 1 above; and it becomes a moral certainty when we come to Problem IV, where Barrow is at last successful.

The first sentence of the opening remark to this Appendix, which I have put in italics, makes it certain that these were Barrow's own work. The reference to Wallis at the end of Problem IV almost "shouts" the fact that it was through reading Wallis' work that Barrow began to accumulate, as was his invariable practice, a collection of *general theorems* connecting an arc with an area; it is also probable that it was only just before publication that he was able to complete his collection with the proof that the area under a hyperbola was a logarithm.

The proof, as Barrow states, for the construction given in Problem 1 is very easily made out, by drawing another ordinate NFQY parallel and near to MEPY and MW parallel to PQ to cut NQ in W. For we have then

$$PE/PA = RS/AR = PY/AR = MW/NW = MP/PT, \therefore AE // MT.$$

Example 1 is not truly an example of the problem; if we allow for Barrow's inversion of the figure (a bad habit of his that probably caused trouble to his readers), to render this a true example of the method of the problem, AD . PM should be made equal to the space DYP instead of the space AYP; this, however, would have made the curve lie on the same side of AD as the quadrant, at *an infinite distance*; so Barrow subtracts the infinite constant, equal to the area QADY, and thus gets a curve lying on the other side of the line AD, fulfilling the required conditions. Example 2, however, is a true example of the problem; and it is particularly noteworthy on account of the fact that it rectifies the cycloid, a result previously attained in

Lect. XII, § 20, Ex. 5; but, as has been noted, the matter is not so clearly put in that as it is here; for, in this Example 2, since $AE^2 = AP . AR$, the curve AEG is evidently a circle, and it follows from the property that the tangent at M is parallel to AE, that the curve AMB is the cycloid; the theorem states that the arc AM is equal to twice the chord AE; and thus Barrow has undoubtedly rectified the cycloid, and thus anticipated Sir C. Wren, who published his work in the *Phil. Trans.* for 1673. Moreover, and Barrow seems to be prouder of this fact than anything else, Barrow's theorem is a general theorem for the rectification of all curves of the form given by

$$X = 2a\, cos^{m/n}\, \theta, \quad Y = 2am/n\, . \int_{\theta}^{0} sin^2\, \theta\, cos^{(m-n)/n}\, \theta\, d\theta.$$

If $m/n = 2$, the curve, as Barrow remarks, is a cycloid; this is also evident analytically if the equations above are worked out. If, however, m/n is equal to any odd integer, the curve AEG has a polar equation $r = a\, cos^{2s}\, \theta$, and the curve AMB is one of the form given by the equations

$$X = a\, cos^{2s+1}\, \theta, \quad Y = a(2s+1)\int_{\theta}^{0} sin^2\, \theta\, cos^{2s-1}\, \theta\, d\theta\,;$$

and this, in the particular case when $s = 1$, is the three-cusped hypocycloid, $X^{2/3} + Y^{2/3} = a^{2/3}$, and the arc of this curve is given as $3AE/2$ (for my n is Barrow's $m - n$), or $3\, a^{1/3} x^{2/3}$; and thus the theorem also rectifies the three-cusped hypocycloid; though, of course, Barrow does not mention this curve, nor can I see a simple theorem by which Barrow could have performed the integration, denoted by $\int sin^2\, \theta\, cos\, \theta\, d\theta$, by a *geometrical construction.*

PROBLEM 2

To draw a curve, AMB say, of which the axis is AD, such that, any point M being taken in it, if MP is drawn perpendicular to AD and MT is supposed to be a tangent to the curve, then TP : PM shall be an assigned ratio.

Let any straight line R be taken; find PY, such that TP : PM (which ratio the assigned relation will give) is equal to the ratio R : PY (and this is to be taken along the line

PM and at right angles to the axis AD); and through the points Y obtained in this way let the curve YYK be drawn; then, if PM is made equal to the space APY/R, the nature of the curve AMB will be established.

Example 1.—Let ADG be a quadrant of a circle, of which the radius is equal to the assigned length R; let it be desired that the ratio of TP to PM shall be equal to that of R to arc AE; then, since as prescribed, R : arc AE = R : PY; PY = arc AE; and hence PM = APY/R.

Example 2.—Let ADG be a quadrant of a circle, and suppose that the ratio TP : PM has to be equal to that of PE : R; then PY will be equal to the tangent of the arc GE; and the space APYY is equal to R . arc AE. Then PM = arc AE.

PROBLEM 3

Being given any figure AMBD whose axis is AD and base DB, it is required to find a curve KZL such that, when any straight line ZPM is drawn parallel to DB, cutting AD in P, and it is supposed that ZT touches the curve KZL, then TP = PM.

The construction is as follows:—

Let OYY be a curve such that, any finite straight line R being taken, and PMY produced, PM : R = R : PY; then, taking any point L in BD produced, draw LE at right angles to DL,* so that DL : R = R : LE; then, with asymptotes DL, DG,* describe the hyperbola EXX passing through E; let the space LEXH be equal to the space DOYP, and produce XH and YP to meet in Z. Then will Z be a point in the required curve, and if ZT is a tangent to it, TP = PM.

---

* AD is produced to G and LE is in the same sense as DG.

It is to be noted that, if the given figure is a rectangle ADBC, the curve KZL has the following property. DH is a geometric mean between DL and DO of the same order as DP is an arithmetic mean between DA and O (or zero).

Now, if any curve KZL is described with this property, and the tangent ZT is found practically, then the hyberbolic space LEXH will be found, and this in all cases is equal to the rectangle contained by TP and AP.*

It can also be seen that

(i) the space ADLK = R(DL − DO);

(ii) the sum of ZP^2, etc. = $R(DL^2 - DO^2)/2$, and the sum of ZP^3, etc. = $R(DL^3 - DO^3)/3$, and so on; †

(iii) if it is supposed that ϕ is the centre of gravity of the figure ADLK, and $\phi\psi$ is drawn perpendicular to AD and $\phi\xi$ to DL, then $\phi\psi = (DL + DO)/4$, and $\phi\xi = R - AD \,.\, DO/LO$.

PROBLEM 4

Let BDH be a right angle, and BF parallel to DH; with DB, DH as asymptotes, let a hyperbola FXG be described to pass through F; with centre D describe the circle KZL; lastly let AMB be a curve such that, if any point M is taken in it, and through M the straight line DMZ is drawn, and it is also assumed that DI = DM and IX is drawn parallel to BF, then the hyperbolic space BFXI is equal to twice the circular sector ZDK. It is required to draw the tangent at M to the curve AMB.

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* Here Barrow seeks *the curve whose subtangent is constant* and obtains it; he, however, does not at first seem to perceive the exponential character of it. For, although he states the property of the geometric and arithmetic means, it is not till in connection with the next problem that he states that this has anything to do with logarithms.

† As usual, these quantities have to be applied to AD.

Draw DS perpendicular to DM, and let $DB . BF = R^2$; then make $DK : R = R : P$, and then $DK : P = DM : DT$; join TM; then TM will touch the curve AMB.

It is to be observed that the curve has the following property. DI is a geometric mean between DB and DO (or DA) of the same order as the arc KZ is an arithmetic mean between 0 (or zero) and the arc KL. That is, if DI is a number in the geometric series beginning with DB and ending with DA, and 0, KL are the logarithms of DB, DA, then KZ will be the logarithm of DI. Or, working the other way (the way in which ordinary logarithms go), if DI is a number in the geometric series starting with DO and ending with DB, and 0 is the logarithm of DO, and the arc LK that of DB, then the arc LZ will be the logarithm of DI.

Now, if the curve is completely drawn and the tangent to it determined practically, it is evident that the circular equivalent of the hyperbolic space is given, or the hyperbolic equivalent of the circular sector.

That most eminent man, Wallis,* worked out most clearly the nature and measurement of this Spiral (as well as of the space BDA) in his book on the cycloid; and so I will say no more about it.

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* Wallis' chief works connected with the problems of Infinitesimal Calculus are in course of preparation, and will be issued shortly; so that it has not been thought necessary to give here anything further than this reference.

NOTE

The two foregoing propositions are particularly interesting in their historical associations. Logarithms had been invented at the beginning of the seventeenth century, and the method of Briggs (*Arithmetica Logarithmica*, 1624) was still fresh. Logarithms were devised as numbers which increased in arithmetical progression as other numbers related to them increased in geometrical progression. We know that Wallis had evaluated the integral of a positive integral power of the variable, and later had extended his work to other powers; Cavalieri had also obtained the same results working in another way; also Fermat had used the method of arithmetic and geometric means as the basis of his work on integration, and he specially remarks that it is a logarithmic method; but it was left to Gregory St Vincent to perform the one remaining integration of a power when the index was -1. This he did by the method of exhaustions, working with a rectangular hyperbola referred to its asymptotes; he stated (in 1647) that, if areas from a fixed ordinate increased in arithmetic progression, the other bounding ordinates decreased in geometric progression.* This is practically identical with the special type that Barrow takes as an example to Problem 3; but it was left to Barrow to give the result in a definite form. At the same time we see that, if Barrow owed anything at all to Fermat, we must credit Fermat's remark with being the source of Barrow's ideas on the application of these arithmetic and geometric means. As usual with Barrow, he gives a pair of theorems, perfectly *general in form*, one for polar and the other for rectangular coordinates. He proves that the area under the hyperbola referred to its asymptotes, included between two ordinates whose abscissae are a, b, is $\log(b/a)$, though he is unaware apparently of the value of the base of the logarithm. I say apparently, because I will now show that it is quite within the bounds of probability that Barrow had found it by calculation;

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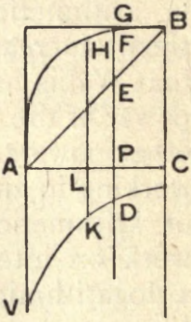
* Brouncker used the same idea in 1668 to obtain an infinite series for the area under a hyperbola.

supposing my suggestion is true, however, Barrow would at that time have been unable to have proved his calculation *geometrically*, or indeed in any other theoretical manner, and so would not have mentioned the matter; as we see, he leaves the constant to be determined *practically* (*Mechanice*), this way being just as good in his eyes as any other that was not geometrical.



Let AFB be a paraboliform such that PF is the first of $m - 1$ means between PG and PE. Also let VKD be another curve such that

$$\text{space AVDP} = \text{R} \,.\, \text{PF}.$$

Then, taking AC = CB, to avoid a constant, the equation to AFB is

$$\text{PF}^m = \text{AP} \,.\, \text{PG}^{m-1},$$

and the equation of VDK is

$$(m \,.\, \text{PD})^m \,.\, \text{AP}^{m-1} = \text{R}^m \,.\, \text{PG}^{m-1}.$$

Now area LKDP = R . (PF – HL)

$$\therefore \text{LP} \,.\, \text{PD} = \text{R} \,.\, \text{PG}^{1-1/m} \,.\, (\text{AP}^{1/m} - \text{AL}^{1/m}).$$

Hence, if we put x for AP, we obtain $\int dx/x^{1-1/m}$ = the sum of $m \,.\, \text{LP} \,.\, \text{PD}/\text{R} \,.\, \text{PG}^{1-1/m} = m \,.\, (\text{AP}^{1/m} - \text{AL}^{1/m})$.

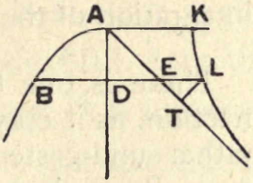
But if m is indefinitely increased, and R is taken equal to m, the curve VKD tends to become a rectangular hyperbola; and in Problems 3, 4, Barrow has shown that the area is proportional to *log* AP/AL. Hence *log* AP/AL is the limiting value of $m(\text{AP}^{1/m} - \text{AL}^{1/m})$, when m is indefinitely increased, or *log* x is the limiting value of $(x^n - 1)/n$, when n is indefinitely small.

Now remembering that Briggs in his *Arithmetica Logarithmica* had given the value of 10 to the power of $1/2^{54}$ as 1·00000000000000012781914932003235, it would not have taken five minutes to work out *log* 10 = 2·3058509 . . .; hence, calling this number μ, Barrow has

$$\int dx/x = \mu \,.\, log_{10}\, x.$$

Considering Barrow's fondness for the paraboliforms, it would seem almost to be impossible that he should not have carried out this investigation; although, if only for his usual disinclination to "put his sickle into another man's harvest," as he remarks at the head of Problem 9, he does not publish it; he in fact refers to Wallis' work on the Logarithmic Spiral as a reason why he should say no more about it. It is to be noted that, in Problem 4, Barrow constructs the Equiangular Spiral, and then proves it to be identical with the Logarithmic Spiral. Hence, if $r\frac{d\theta}{dr} = \mathrm{C}$, $r = a^\theta$ and conversely; thus $d(a^x)/dx = \mathrm{K}a^x$ and $\int a^x dx = ma^x$, where K, m have to be determined.

If we do not allow that Barrow had found out a value for the base of the logarithm, yet assuming *log x* to stand for a logarithm to an unknown base, Barrow has rectified the *parabola*, effected the integration of *tan θ*, and the areas of many other spaces that he has reduced to the quadrature of the hyperbola. For instance, in Lect. XII, § 20, Ex. 3, he shows that Z . arc AB = area ADLK.



$$\text{Now} \qquad \mathrm{ATLK} = \mathrm{Z}^2 . \log(\sqrt{2} . \mathrm{AT}/\mathrm{Z})/2 + \mathrm{Z}^2/4$$

$$\therefore \mathrm{ADLK} = \tfrac{1}{2}\mathrm{Z}^2 . \log(\mathrm{AD}+\mathrm{DL})/\mathrm{Z} + \tfrac{1}{2}\mathrm{AD} . \mathrm{DL}.$$

In modern notation, since $\mathrm{Z} . \mathrm{DL} = \mathrm{AD}^2/2$,

$$\int_0^x \sqrt{(x^2+a^2)}dx = \tfrac{1}{2}a^2 . \log[\{x+\sqrt{(x^2+a^2)}\}/a] + \tfrac{1}{2}x . \sqrt{(x^2+a^2)},$$

where the base of the logarithm has to be determined.

Similarly, in Lect. XII, App. I, § 2, he states that the sum of the tangents belonging to the arc AM applied to the line $a\mu$ is equal to the hyperbolic space AFZK; that is,

$$\int_0^\theta \tan\theta\, d\theta = \log \mathrm{AF}/\mathrm{AC} = \log\cos\theta.$$

The theorem of § 1 is the same thing in another form.

Again, in Lect. XII, App. I, § 4, we have the equivalent of the integral of *sin θ*; since Barrow's integrals are all definite, we find it in the form $\int_0^\theta \sin\theta\, d\theta = 2\cos^2\theta/2$.

From § 5, we obtain $\int_0^\theta \sec^2\theta\, d(\sin\theta) = \int_0^\theta d\theta/\cos\theta$ or

$\int_0^\theta \sec\theta\, d\theta$ given as $\frac{1}{2} \log\{(1+\sin\theta)/(1-\sin\theta)\}$, which of course can be reduced immediately to the more usual form $\log\tan(\theta/2+\pi/4)$; the same result is obtained from §§ 6, 7; or they can be exhibited in the form $\int dx/(a^2-x^2) = \{\log(a+x)/(a-x)\}/2a$.

The theorem of § 8 is a variant of the preceding and proves that $\int \cos\theta\; d(\tan\theta)$ is equal to $\int \tan\theta\; d(\cos\theta) - \tan\theta \cdot \cos\theta$, both being equal to $\int \sec\theta\, d\theta$.

The theorem of § 9 reduces immediately to $\int dx/\sqrt{(x^2+a^2)} = \log\{x+\sqrt{(x^2+a^2)}/a\}$.

Thus Barrow completes the usual standard forms for the integration of the circular functions.

There is one other point worth remarking in this connection, as it may account for the rushing into print of this rather undigested Appendix; I have already noted that, from Barrow's own words, this Appendix was added only just before the publication of the book. I imagine this was due to Barrow's inability to complete the quadrature of the hyperbola to his rather fastidious taste. But, in 1668, Nikolaus Kaufmann (*Latine* Mercator) published his *Logarithmotechnia*, in which he gave a method of finding true hyperbolic logarithms (not *Napierian* logarithms); of this publication Prof. Cajori says:—"Starting with the grand property of the rectangular hyperbola . . ., he obtained a logarithmic series, which Wallis had attempted but failed to obtain." (Rouse Ball attributes the series to Gregory St Vincent.) This may have settled any qualms that Barrow had concerning the unknown base of his logarithms, and decided him to include this batch of theorems, depending solely on the quadrature of the hyperbola, and merely requiring a definite solution of the latter problem to enable Barrow to complete his standard forms. Kaufmann obtained his series by shifting one axis of his hyperbola, so that the equation became $y = 1/(1+x)$, expanded by simple *division*, and integrated the infinite series term by term, thus obtaining the area measured from the ordinate whose length was unity, and avoiding the infinite area close to the asymptote.

PROBLEM 5

Let EDG be any space bounded by the straight lines DE, DG and the curve ENG, and R any straight line of given length; it is required to find a curve AMB such that, when any straight line DNM is drawn from D, and DT is perpendicular to it, and MT touches the curve AMB, then shall DT : DM = R : DN.*

Let KZL be a curve such that $DZ^2 = R . DN$, and, the straight line DB being drawn, of arbitrary length, let DB : R = R : BF, where BF, and also DH, is at right angles to DB. Then through F, within the angle BDH, draw the hyperbola FXX, and let the space BFXI (where IX is supposed to be parallel to BF) be equal to double the space ZDL; lastly, let DM = DI. Then M will be a point on a curve such as is required; and if a straight line MT touches the curve at any point M, then will TD : TM = R : DN.

PROBLEM 6

Again, let EDG be a given space (as in the preceding); it is required to find a curve AMB such that, if any straight line DNM is drawn, and DT is perpendicular to it, and MT touches the curve, then DT shall be equal to DN.

Take any straight line of length R, and let $DZ^2 = R^3/DN$; also having taken DB (to which DH, and BF ($= R^3/DB^2$), are perpendiculars) assume that through F is drawn, between the asymptotes DB, DH, a hyperboliform of the second kind (that is, one in which the ordinates, as BF or IX, are fourth proportionals in the ratio DB to R,* or DI to R). Then take the space BIXF equal to double the space ZDL; and let $DM = DI$; then M will be a point on the required curve; and if MT touches it, $DT = DN$.

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* The next four problems constitute Barrow's conclusion of his work on Integration. Probs. 5, 6 give *graphical* constructions for integration, and 7, 8 find *graphically* the bounding ordinate or radius vector for a figure of given area, *i.e.* graphical differentiation of a kind.

PROBLEM 7

Let ADB be any figure, of which the axis is AD and the base is DB, and, any straight line PM being drawn parallel to DB, let the space APM be given (or expressed in some way); it is required from this to draw the ordinate PM, or to give some expression for it.

Take any straight line R, and let $R \cdot PZ =$ space APM; in this way let the line AZZK be produced; find ZO the perpendicular to it; then $PZ : PO = R : PM$.

Otherwise. Take $PZ = \sqrt{(2APM)}$; let ZO be perpendicular to the curve AZK; then $PM = PO$.

PROBLEM 8

Let ADB be any figure, bounded by the straight lines DA, DB and the curve AMB, and through D let any straight line DM be drawn; given the space ADM, it is required to find the straight line DM.

Take any straight line R, and let $DZ = 2ADM/R$; draw ZO perpendicular to the curve AZK; let DH, the perpendicular to DM, meet it; then $DM^2 = R \cdot DO$.

Otherwise. Let $DZ = \sqrt{(4ADM)}$; and draw ZO perpendicular to the curve AZK; let DH, the perpendicular to DZ, meet it; then $DM^2 = DZ \cdot DO$.

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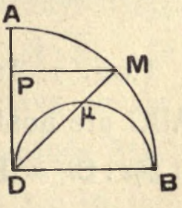
* If $DB : R = R : P = P : BF$, $DB^2 : R^2 = R : BF$, or $BF = R^3/DB^2$.

NOTE

These four problems are generally referred to by the authorities as "inverse-tangent" problems. I do not think this was Barrow's intention. They are simply the completion of his work on integration, giving as they do a method of integrating *any function*, which he is unable to do by means of his rules, by *drawing and calculation*. Thus, the problem of § 5 reduces to:—"Given any function, $f(x)$ say, construct the curve whose polar equation is $r = f(\theta)$, perform the given construction, and the value of $\int_0^x f(x)dx$ is equal to R *log* DB/DI or R *log* DB/DM." Similarly, in Problem 6, the value of $\int dx/f(x)$ is given as 1/DM - 1/DB. The construction as given demands the next two problems, or one of those which follow, called by Barrow "evolute and involute" constructions. As an alternative, Barrow gives an envelope method by means of the sides of the polar tangent triangle. It is rather remarkable that as Barrow had gone so far, he did not give the mechanical construction of derivative and integral curves in the form usual in up-to-date text-books on practical mathematics, which depend solely on the property that differentiation is the inverse of integration.

With regard to the propositions that follow under the name of problems on "evolutes and involutes," it must be noted that, although at first sight Barrow has made a mistake, since the involute of a circle is a spiral and cannot under any circumstances be a semicircle; yet this is not a mistake, for Barrow's definition of an involute (whether he got it from James Gregory's work or whether he has misunderstood Gregory) is not the usual one, but stands for a polar figure equivalent in area to a given figure in rectangular coordinates, and *vice versa*. In a sense somewhat similar to this Wallis proves that the circular spiral is the involute of a parabola.



Hence, these problems give alternative methods for use in the given constructions for Problems 5, 6. Thus in the adjoining diagram, it is very easily shown that area D$\mu\mu$B = $\frac{1}{2}$ area DBMP.

That brilliant geometer, Gregory of Aberdeen, has set on foot a beautiful investigation concerning involute and evolute figures. I do not like to *put my sickle into another man's harvest*, but it is permissible to interweave amongst these propositions one or two little observations pertaining in a way to such curves, which *have obtruded themselves upon my notice whilst I have been working at something else.*

PROBLEM 9

Let ADB be any given figure, of which the axis is AD and the base is DB; it is required to draw the evolute corresponding to it.

With centre C, and any radius CL, let a circle LXX be described; also let KZZ be a curve such that, when any line MPZ you please is drawn parallel to BD, the rectangle contained by PM and PZ is equal to the square on CL (or PZ is equal to CL^2/PM). Then let the arc LX = space DKZP/CL (or sector LCX = half the space DKZP) and in CX take $C\mu = PM$; then the line $B\mu\mu$ is the involute of BMA, or the space $C\mu\beta$ of the space ADB.

For instance, if ADB is a quadrant of a circle, the line $\beta\mu C$ is a semicircle.

COR. 1. It is to be observed that if the two figures ADB, ADG are analogous; and the involutes are $C\mu\beta$, $C\nu\gamma$; and if $C\mu : C\nu = DB : DG$; then, reciprocally,

$$\angle \beta C\mu : \angle \beta C\nu = DG : DB.$$

COR. 2. The converse of this is also true.

COR. 3. If $C\nu\gamma$, $CS\beta$ are analogous *suo modo*, that is if, when any straight line $C\nu S$ is drawn through C, $C\nu$ to CS is always in the same ratio; then these will be the involutes of similar lines.

PROBLEM 10

Given any figure $\beta C\phi$, bounded by the straight lines $C\beta$, $C\phi$, and another line $\beta\phi$; it is required to draw the evolute.

With centre C, describe any circular arc LE (making with the straight lines $C\beta$, $C\phi$ the sector LCE); then, CK being drawn perpendicular to LC, let the curve βYH be so related to the straight line CK that, when any straight line $C\mu Z$ is drawn, and CO is taken equal to the arc LZ, and OY is drawn perpendicular to CK, $OY = C\mu$. Also, let the curve BMF be so related to the straight line DA that, when DP is equal to space $C\beta YO/CL$, and PM is drawn perpendicular to DA, then $PM = C\mu$ also. Then the space $DBFA$ is the evolute of $C\beta\phi$.

Example.—Let LZE be the arc of a circle described with centre C, and $\beta\mu C$ a spiral of such a kind that, if the straight line $C\mu Z$ is drawn in any manner, the arc EZ always bears to the straight line $C\mu$ some assigned ratio (say, $R : S$). It is plain that the line βYH is straight, for we have always EZ (or KO) : $C\mu$ (or OY) $= R : S$. Hence, the evolute BMF is a parabola, since the parts AP, AD of the axis are in the same ratio as the spaces KOY, $KC\beta$, that is, as the squares on OY, $C\beta$, or the squares on PM, DB.

Corollaries

Theorem 1. If on the figure $\beta C\phi$ is erected a cylinder having its altitude equal to the whole circumference of the circle whose radius is CL; then the cylinder will be equal to the solid produced by rotating the figure $C\beta HK$ about CK.

Theorem 2. Let AMB be any curve of which the axis is AD and the base is DB, and AZL a curve such that, when any straight line ZPM is drawn, $PZ = \surd(2APM)$; and let OYY be another curve such that, when the straight line ZPMY is produced to meet it, $ZP^2 : R^2 = PM : PY$. Lastly, let $DL : R = R : LE$, and through E, within the angle LDG, describe the hyperbola EXX; let the straight line ZHX, drawn parallel to AD, meet it in X. Then the space PDOY will be equal to the hyperbolic space LXHE.

Hence, the sum of all such as $PM/APM = 2LEXH/R^2$.

Theorem 3. Let AMB be any curve whose axis is AD and base is DB; and let the curve KZL be such that, if any straight line R is taken, and an arbitrary line ZPM is drawn parallel to BD, $\surd APM : PM = R : PZ$; then the space ADLK is equal to the rectangle contained by R and $2\surd ADB$, or $ADLK/2R = \surd ADB$.*

Example.—Let ADB be a quadrant of a circle; then the sum of all such as $PM/\surd APM = \surd(2DA \,.\, \text{arc } AB)$.

Theorem 4. Let AMB be any curve of which the axis is AD and the base is DB, and let EXK, GYL be two lines so related that, any point M being taken in the curve, and the straight lines MPX, MQY being drawn respectively parallel to BD and AD, and it being supposed that MT touches the curve AMB, then TP : PM = QY : PX. Then will the figures ADKE, DBLG be equal to one another.

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* The theorems equivalent to Theorems 2 and 3 are clear enough, even without the final line in the first of the pair; Barrow intends them as standard forms in integration.

NOTE. Of all the propositions so far, this theorem is the most fruitful; since many of the preceding are either contained in it or can be easily deduced from it. For, suppose the line AMB is by nature indeterminate, then if one or other of the curves EXK, GYL is determined to be anything you please, there will result from the supposition some theorem of the kind of which we have given a considerable number of examples already. If, for instance, the line GYL is supposed to be a straight line making with BD an angle equal to half a right angle (in which case the points D, G are taken to be coincident), then we get the theorem of Lect. XI, § 1.

If GYL is a line parallel to DB, we have Lect. XI, § 11.

Again, if PM = PX (or the lines AMB, EKX are exactly the same), hence follows Lect. XI, § 10.

Further it is plain from the theorem that for any given space an infinite number of equal spaces of a different kind can be easily drawn; thus, if the space DGLB is supposed to be a quadrant of a circle, centre D, and AMB is a parabola whose axis is AD, we get this property of the curve EKX (by putting DB = r, AP = x, PX = y, and k for the semi-parameter of the parabola or DB^2/2AB), that $r^2k/2 = k^2x + xy^2$.

If, however, AMB is supposed to be a hyperbola, there will be produced a curve EXK of another kind.

Moreover, on consideration, I blame my lack of foresight, in that I did not give this theorem in the first place (it and those that follow, of which the reasoning is similar and the use almost equal); and then from it (and the others that are added directly below), as I see can be done, have deduced the whole lot of the others. Nevertheless, I think that this sort of Phrygian wisdom is not unknown either to me or to others who may read this volume.

NOTE

When I first considered the title-pages of the volume from which I have made the translation, I was struck by the fact that the *Lectiones Opticæ* had directly beneath the main title the words "*Cantabrigiæ in Scholis publicis habitæ*" (delivered in the public Schools of Cambridge), whereas no such notification appears on the separate title-page of the *Lectiones Geometricæ*. When later I found that the title-page of the *Lectiones Mathematicæ* also bore this notification, I became suspicious that at any rate there was *no direct evidence* that these lectures on Geometry had ever been delivered as professorial lectures, though they might have been given to his students by Barrow in his capacity of college fellow and lecturer. As I considered the Preface, I was confirmed in this opinion; and the above note would seem to corroborate this suggestion. For surely if these matters had been given in University Lectures in the Schools, it would not have been necessary to wait till they were ready for press before Barrow should find out that his Theorem 4 was more fruitful and general than all the others. His own words contradict the supposition that he initially did not know this theorem, for he blames his "want of foresight." This raises the point as to the exact date when Newton was shown these theorems; this has been discussed in the Introduction.

Theorem 5. Let ADB be any space, bounded by the straight lines DAE, DQBK and the curve AMB, also let EXK, GYL be two curves so related that, if any point M is taken in the curve AMB, and DMX is drawn, and DQ = DM, and QYBL, DG are drawn perpendicular to DB, and DT is perpendicular to DM, and the straight line MT touches the curve AMB; if, I say, when these things are so, $TD : DM = DM . QY : DX^2$; then shall the space DGLB be double the space EDK.

Theorem 6. Again, let AMB be any curve of which the axis is AD and the base is DB; and let EXK, HZO be two curves so related to one another and the axes AD, $\alpha\beta$ so related to one another that, if a point M is taken anywhere on the curve AMB, and MPX is drawn perpendicular to AD, and $\alpha\mu$ is taken equal to AM, and μZ is drawn perpendicular to $\alpha\beta$, and it is supposed that MT touches the curve AMB, and cuts DA in T, then $TP : PM = \mu Z : PX$. Then the spaces ADKE, $\alpha\beta OH$ shall be equal to one another.

Theorem 7. Let ADB be any space, bounded by the straight lines DAE, DBK and the curve AMB; also let EXK, HZO be two curves so related that, if any point M is taken in the curve AMB, and the straight line DMX is drawn, and $\alpha\mu$ is taken equal to the arc AM, and μZ is drawn perpendicular to the straight line $\alpha\beta$, and DT is perpendicular to DM, and the straight line MT touches the curve AMB, then $DT : DM = DM . \mu Z : DX^2$. Then shall the space $\alpha\beta OH$ be double the space EDK.

But here is the end of these matters.

LECTURE XIII

[The subject of this lecture is a discussion of the roots of certain series of connected equations. These are very ingeniously treated and are exceedingly interesting, but have no bearing on the matter in hand; accordingly, as my space is limited, I have omitted them altogether.]

Laus DEO *Optimo Maximo*

FINIS

* * *

In the second edition, published in 1674, there were added the three problems given below, together with a set of theorems on Maxima and Minima. Problem II is very interesting on account of the difficulty in seeing how Barrow arrived at the construction, unless he did so *algebraically*.

PROBLEM I. Let any line AMB be given (of which the axis is AD, and the base DB), it is required to draw a curve ANE, such that if any straight line MNG is drawn parallel to BD, cutting ANE in N, then the curve AN shall be equal to GM.

The curve ANE is such that if MT touches the curve AMB, and NS the curve ANE, then $SG : GN = TG : \sqrt{(GM^2 - TG^2)}$.

PROBLEM II. With the rest of the hypothesis and construction remaining the same, let now the curve ANE be required to be such that the arc AN shall be always equal to the intercept MN.

Let the curve ANE be such that $SG : GN = 2TG \cdot GM : (GM^2 - TG^2)$, then ANE will be the required curve.

PROBLEM III. Let any curve DXX be given, whose axis is DA; it is required to find a curve AMB with the property that, if any straight line GXM is drawn perpendicular to AD, and it is given that SMT is the tangent to the curve AM, then $MS = GX$.

Clearly the ratio $TG : TM$ (that is, the ratio of GD to MS or GX) is given; and thus the ratio $TG : GM$ is also given.

Barrow does not give proofs of these problems. The only geometrical proof of the second I can make out is as follows:—

Draw PQR parallel to GNM, cutting the curves ANE, AMB in Q, R respectively, and draw MW, NV parallel to AD to meet PQR in W, R. Then we have $NQ = RW - QV$ from the supposed nature of the curve; also from the several differential triangles, we have $RW/GP = MG/GT$, $QV/GP = NG/GS$, and $NQ/GP = NS/SP$; and therefore

$$NS \cdot GT = MG \cdot GS - GT \cdot NG.$$

Squaring,

$$(NG^2 + GS^2) \cdot GT^2 = MG^2 \cdot GS^2 - 2MG \cdot GS \cdot GT \cdot GN + GT^2 \cdot NG^2;$$

hence, $$GS \cdot (GM^2 - GT^2) = 2MG \cdot GT \cdot NG,$$

or $$GS : GN = 2MG \cdot GT : (GM^2 - GT^2).$$

But I can hardly imagine Barrow performing the operation of squaring, unless he was working with algebraic symbols; in this case he would be using his theorem that

$$(ds/dx)^2 = 1 + (dy/dx)^2. \quad \text{(Lect. X, § 5.)}$$

POSTSCRIPT

Extracts from Standard Authorities

Since this volume has been ready for press, I have consulted the following authorities for verification or contradiction of my suggestions and statements.

ROUSE BALL (*A Short Account of the History of Mathematics*)

(i). "It seems probable, from Newton's remarks in the fluxional controversy, that Newton's additions were confined to the parts" (of the *Lectiones Opticæ et Geometricæ*) "which dealt with the Optics."

(ii). "The lectures for 1667 . . . suggest the analysis by which Archimedes was led to his chief results."

(iii). "Wallis, in a tract on the cycloid, incidentally gave the rectification of the semi-cubical parabola in 1659; the problem having been solved by Neil, his pupil, in 1657; the logarithmic spiral had been rectified by Torricelli" (*i.e.* before 1647). "The next curve to be rectified was the cycloid; this was done by Wren in 1658."

(This contradicts Williamson entirely; I suggest that, of the two, Ball is probably the more correct, if only for the fact that this would explain why Barrow did not remark on the fact that he had rectified both the cycloid and the logarithmic spiral.)

(iv). The only thing in Barrow's work that is given any special notice is the differential triangle; since Ball states that his great authority for the time antecedent to 1758 is M. Cantor's monumental work *Vorlesungen über die Geschichte der Mathematik*, it would appear that Cantor also does not give Barrow the credit that he deserves.

(v). Fermat had the approximation to the binomial theorem; for he was able to state that the limit of $e/\{1-(1-e)^{5/3}\}$, when e is evanescent, is 3/5. Since we know that Fermat had occupied himself with arithmetic and geometric means, it would seem probable that Barrow's equivalent theorem was deduced from this work of Fermat; however, Ball states that these theorems of Fermat were not published until after his death in 1665, whereas Barrow's theorem was, I have endeavoured to show, considerably anterior to this.

(vi). With reference to the Newton controversy we have :—
"It is said by those who question Leibniz' good faith, that to a man of his ability the manuscript (Newton's *De Analysi per Aequationes*), especially if supplemented by the letter of Dec. 10, 1672, would supply sufficient hints to give him a clue to the methods of the calculus, though as the fluxional notation is not employed in it, anyone who used it would have to invent a notation."

(How much more true is this of Barrow's Lectures, which contained a complete set of standard forms and rules, and was much more like Leibniz' method, in that it did not use series but gave rules that would work through *substitutions*! See under Gerhardt.)

"Essentially it is Leibniz' word against a number of suspicious details pointing against him."

(I hold that the dates are almost conclusive, as they are given in the fourth paragraph of the preface; and in this I do not by any means suggest that Leibniz lied, as will be seen under Gerhardt. A mathematician, having Leibniz' object and point of view, would more probably consider that Barrow's work and influence was a hindrance rather than a help, *after* he had absorbed the fundamental ideas.)

Professor Love (*Encyc. Brit. XIth. ed., Art.* "*Infinitesimal Calculus.*")

(i). "Gregory St Vincent was the first to show the connection between the area under the hyperbola and logarithms, though he did not express it analytically. Mercator used the connection to calculate natural logarithms."

(ii). "Fermat, to differentiate irrational expressions, first of all rationalized them; and although in other works he used the idea of substitution, he did not do so in this case."

(iii). "The *Lectiones Opticæ et Geometricæ* were apparently written in 1663–4."

(iv). "Barrow used a method of tangents in which he compounded two velocities in the direction of the axes of x and y to obtain a resultant along the tangent to a curve."

"In an appendix *to this book* he gives another method which differs from Fermat's in the introduction of a second differential."

(Both these statements are rather misleading.)

(v). "Newton knew to start with in 1664 all that Barrow knew, and that was practically all that was known about the subject at that time."

(vi). "Leibniz was the first to differentiate a logarithm and an exponential in 1695."

(Barrow has them both in Lect. XII, App. III, Prob. 4.)

(vii). "Roger Cotes was the first, in 1722, to differentiate a trigonometrical function."

(It has already been pointed out that Barrow explicitly differentiates the tangent, and the figures used are applicable to the other ratios; he also integrates those of them which are not thus obtainable by his inversion theorem from the differentiations. Also in one case he integrates an inverse cosine, though he hardly sees it as such. With regard to the date 1722, as Professor Love kindly informed me on my writing to him, this is the date of the posthumous publication of Cotes' work; Professor Love referred me to the passages in Cantor from which the information was obtained.)

(viii). "The integrating curve is sometimes referred to as the Quadratrix."

(This is Leibniz' use of the term, and not Barrow's. With Barrow, the Quadratrix is the particular curve whose equation is

$$v = (r - x)\,\tan \pi x/2r.\ )$$

There are a host of other things both in agreement with and in contradiction of my statements to be found in this erudite article; nobody who is at all interested in the subject should miss reading it. But I have only room for the few things that I have here quoted.

Dr GERHARDT (*Editions of Leibnizian Manuscripts, etc.*)

(i). In a letter to the Marquis d'Hopital, Leibniz writes:—"I recognize that Barrow has gone very far, but I assure you, Sir, that I have not got any help from his methods. As I have recognized publicly those things for which I am indebted to Huygens and, with regard to infinite series, to Newton, I should have done the same with regard to Barrow."

(In this connection it is to be remembered, as stated in the Preface, that Leibniz' great idea of the calculus was the freeing of the work from a geometrical figure and the convenient notation of his calculus of differences. Thus he might truly have received no help from Barrow in his estimation, and yet might, as James Bernoulli stated in the *Acta Eruditorum* for January 1691, have got all his fundamental ideas from Barrow. Later Bernoulli (*Acta Eruditorum*, June 1691) admitted that Leibniz was far in advance of Barrow, though both views were alike in some ways.)

(ii). Leibniz (*Historia et Origo Calculi Differentialis*) states that he obtained his "characteristic triangle" from some work of Pascal (*alias* Dettonville), and not from Barrow. This may very probably be the case, if he has not given a wrong date for his reading of Barrow, which he states to have been 1675; this would not seem to be an altogether unprecedented proceeding on the part of Leibniz, according to Cantor. It is difficult to imagine that Leibniz, after purchasing a copy of Barrow on the advice of Oldenburg, especially as in a letter to Oldenburg of April 1673 he mentions the fact that he has done so, should have put it by for two whole years; unless his geometrical powers were not at the time equal to the task of finding the hidden meaning in Barrow's work.

(iii). Gerhardt states that he has seen the copy of Barrow referred to in the Royal Library at Hanover. He mentions the fact that there are in the margins notes written in Leibniz' own notation, including the sign of integration. He also lays stress on the fact that opposite the Appendix to Lect. XI there are the Latin words for "knew this before." This tells against Leibniz, and not for him, for this Appendix refers to the work of Huygens, which of course Leibniz "knew before," and Gerhardt does not state that *these words occur in any other connection*; hence we may argue that this particular section was the only one that Leibniz "knew before." The sign of integration, though I cannot find any mention of it before 1675, means nothing, for it might be added on a second reference, after Leibniz had found out the value of Barrow's book. A striking "coincidence" exists in the fact that the two examples that Leibniz gives of the application of his calculus to geometry are both given in Barrow. In the first, the figure (on the assumption that it was taken from Barrow) has been altered in every conceivable way; for the second, a theorem of Gregory's quoted by Barrow, Leibniz gives no figure, and *it was only after reference to Barrow's figure that I could complete Leibniz' construction* from the verbal directions he gave. This looks as if Leibniz wrote with a figure beside him that *was already drawn*, possibly in a copy of Gregory's work, or, as I think, from Barrow's figure. I have been unable to ascertain the date of publication of this theorem by Gregory, or whether there was any chance of its getting into the hands of Leibniz in the original.

Professor Zeuthen (*Geschichte der Mathematik im XVI. und XVII. Jahrhundert*; Deutsche Ausgabe von Raphael Mayer).

(i). Oxford and Cambridge seem to be mixed up in the historical section, for it is stated that Barrow was Professor of Greek at Oxford and Wallis was the Professor of Mathematics at Cambridge, as the context suggests that he was Barrow's tutor.

(ii). ". . . he produced his important work, the *Lectiones Mathematicæ*, a continuation of the *Lectiones Opticæ*; this was published, with the assistance of Collins, the first edition in 1669–70, the second edition in 1674."
(Thus Williamson's error is repeated; it would be interesting to know whether Zeuthen and Williamson obtained this from a common source, and also what that source was.)

(iii). "He" (Leibniz) "utilized his stay" (in London in 1673) "to procure the *Lectiones* of Barrow, which Oldenburg had brought to his notice." (See under Gerhardt.)

(iv). Zeuthen, most properly, directs far more attention to the inverse nature of differentiation and integration, as proved by Barrow, than to the differential triangle. But, by his repeated reference to the problem of Galileo, he does not seem to have perceived the fact that the first five lectures were added as supplementary lectures. Yet he notes the fact that Barrow does not adhere to the kinematical idea in the later geometrical constructions. He also calls attention to the generality of Barrow's proofs.

(v). He mentions the differentiation of a quotient, as given in the integration form in Lect. XI, but appears to have missed the fact that the rules for both a product and a quotient have been given implicitly in an earlier lecture.

I have not room for further extracts; each reader of this volume should also read *Zeuthen*, pp. 345–362, if he has not already done so. What he finds there will induce him to read carefully the whole of this excellent history of the two centuries considered.

EDMUND STONE (*Translation of Barrow's Geometrical Lectures, pub. 1735*)

This translation is more or less useless for my purpose. First of all, it is a mere translation, without commentary of any sort, and without even a preface by Stone.

The title-page given states that the translation is "from the Latin edition revised, corrected and amended by the late Sir Isaac Newton." If this refers to the edition of 1670, Stone is in error. But, since at the end of the book, there is an "Addenda," in which are given several theorems that appeared in the second edition, it must be concluded that Stone used the 1674 edition. It is to be remarked that these theorems are on maxima and minima, and, according to the set given by Whewell, only form a part of those that were in the second edition of Barrow; some two or three very interesting geometrical theorems being omitted; one of these is extremely hard to prove by Barrow's methods, and one wonders how Barrow got his theorem; *but the proof "drops out" by the use of dy/dx*, which may account for Barrow having it, but not for Stone omitting it. This seems to give a clue as well to an altogether unjustifiable omission, by either Newton or Stone (I do not see how it could have been Newton, however) at the end of the Appendix to Lect. XI. Two theorems have been omitted; their inclusion was only necessary to prove a third and final theorem of the Appendix as it stood in the first edition; namely, that if CED, CFD are two circular segments having a common chord CGD, and an axis GFE, then the ratio of the seg. CED to the seg. CFD is greater than the ratio of GE to GF. In Stone the two lemmas are omitted and the theorem is directly *contradicted*. The proof given in Stone depends on unsound reasoning equivalent to:—

$$\text{If } a > b, \quad \text{then} \quad c+a:d+b > c:d,$$

without any reference to the value of the ratio of c to d, as compared with that of a to b. Finally the theorem as originally given is correct, as can be verified by drawing and measurement, analytically, or geometrically.

In addition to this alteration, in Stone there is added a passage that does not appear in the first edition, nor is it given in Whewell's edition. "But I seem to hear you crying out 'ἅλλην δρῦν βαλανιζε, Treat of something else.'" In a table of errata the last four words are altered to "Give us something else." The Greek (there should be no aspirate on the first word) literally means "Shake acorns from another oak." If this alteration was made by Stone, the addition of the passage, after the manner of Barrow, is an impertinence. The point is not, however, very important in itself, but taken with other things, points out the comparative uselessness of Stone's translation as a clue to important matters.

The whole thing seems to have been done carelessly and hastily; there hardly seems to have been any attempt to render the Latin of Barrow into the best contemporary English; and frequently I do not agree with Stone's rendering, a remark which may unfortunately cut either way.

Of course the passage may, though it is hardly likely, have been added by Barrow; such an unimportant statement would hardly have been added in those days of dear books; it is also to be noted that Whewell does not give it. The point could only be settled on sight of the edition from which Stone made his translation. Barrow, however, makes a somewhat similar mistake with ratios in Lect. IX, § 10, and Stone passes this and even renders it wrongly. This error has been noted on page 107; the wrong rendering is as follows:—Barrow has $FG : EF + TD : RD$, by which, according to his list of abbreviations, he means $(FG/EF) . (TD/RD)$; and not, as Stone renders it, $FG/EF + TD/RD$, without noticing that this does not make sense of the proof.

Perhaps one sample of the carelessness with which the book has been revised will suffice: he has

$$A \times B = A \text{ dividend } (\textit{sic}) \text{ by } B$$

$$\frac{A}{B} = A \text{ multiplied or drawn into } B;$$

in any case want of space forbids further examples.

It is this untrustworthiness that make it impossible to take Stone's statement on the title-page as incontrovertible; nor another statement that these lectures on geometry were delivered as Lucasian Lectures; it is also to be noted that he gives as Barrow's Preface the one already referred to in the Introduction as the Preface to the Optics and omits the Preface to the Geometry.

WHEWELL (*The Mathematical Works of Isaac Barrow, Camb. Univ. Press*, 1860)

(i). Stress is only laid on two points; one of course is the differential triangle; the other is the "mode of finding the areas of curves by comparing them with the sum of the inscribed and circumscribed parallelograms, leading the way to Newton's method of doing the same, given in the first section of the *Principia*."

(ii). "It is a matter of difficulty for a reader in these days to follow out the complex constructions and reasonings of a mathematician of Barrow's time; and I do not pretend that I have in all cases gone through them to my satisfaction." (This is proof positive that Whewell did not grasp the inner meaning of Barrow's work; that being done, there is, I think, no difficulty at all.)

(iii). The title-page of the *Lectiones Mathematicæ* states that these lectures were the lectures delivered as the Lucasian Lectures in 1664, 1665, 1666; and Lect. XVI. is headed

MATHEMATICI PROFESSORIS LECTIONES
(A.D. MDCLXVI).

(iv). Lect. XXIV starts the work on the method of Archimedes, which would thus appear to be the lectures for 1667, as guessed by me, and as stated by Ball.

(v). Whewell gives the additions that appeared in the second edition of 1674. These consist of four theorems and a group of propositions on Maxima and Minima. One theorem is noteworthy, as its proof depends on the addition rule for differentiation and the fact that

$$(ds/dx)^2 = 1 + (dy/dx)^2.$$

APPENDIX

I. Solution of a Test Question on Differentiation by Barrow's Method

II. Graphical Integration by Barrow's Method

III. *Specimen Pages and Plate*

I. Test Problem suggested by Mr Jourdain

Given any four functions, represented by the curves $\phi\phi$, $\theta\theta$, $\xi\xi$, $\zeta\zeta$, *and given their ordinates and subtangents for any one abscissa, it is required to draw the tangent for this abscissa to the curve whose ordinate is the sum (or difference) of the square root of the product of the ordinates of the first two curves and the cube root of the quotient of the ordinates of the other two curves.*

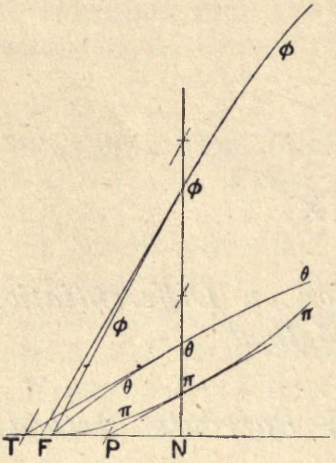
In other words, differentiate

$$\sqrt{\{\phi(x) \,.\, \theta(x)\}} \pm \sqrt[3]{\{\xi(x)/\zeta(x)\}}.$$

The figures on the following page have been drawn for

$$y = \sqrt{\{\sin x \,.\, \log_{10}(\cos x)\}} \pm \frac{1}{10}\sqrt[3]{\{3 \tan x/x^2\}}.$$

(i). Let $N\theta\phi$ be the ordinate for the given abscissa, ϕF, θT the given subtangents; let $\pi\pi$ be a curve such that $R \,.\, N\pi = N\phi \,.\, N\theta$; find NP, a fourth proportional to $NF+NT$, NF, NT; then $P\pi$ will touch the curve $\pi\pi$. [See note on page 112, rule (i).]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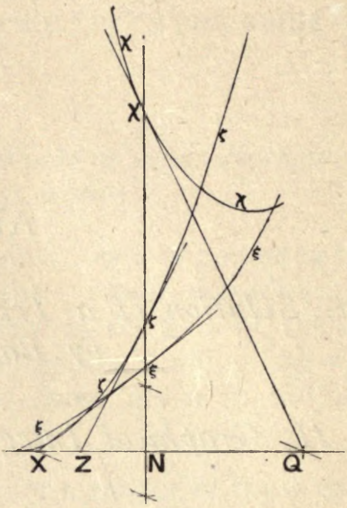
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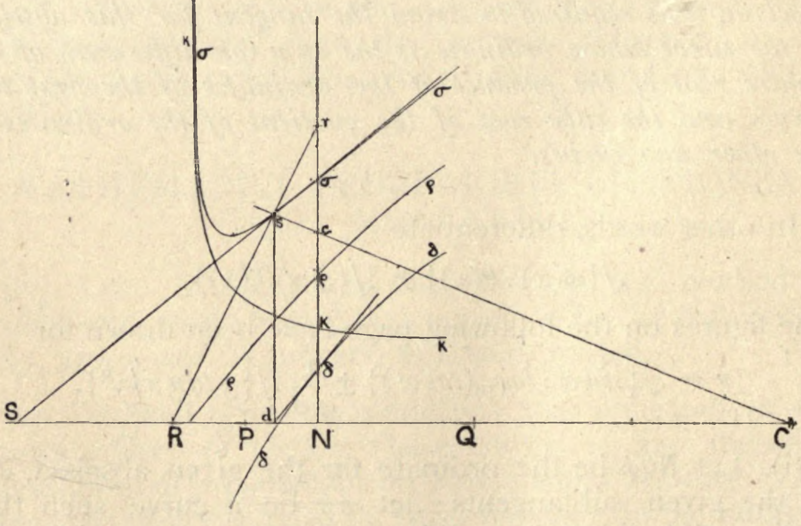
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(ii). Let $N\xi\zeta$ be the ordinate for the given abscissa, ξX, ζZ the given subtangents; let $\chi\chi$ be a curve such that $N\chi : R = N\xi : N\zeta$; find NQ, a fourth proportional to $NZ - NX$, NZ, NX; then $Q\chi$ will touch the curve $\chi\chi$. [See note on page 112, rule (ii).]

(iii). Let $\rho\rho$ be a curve whose ordinate varies as the square root of the ordinate of $\pi\pi$; then its subtangent $NR = 2NP$ (page 104).

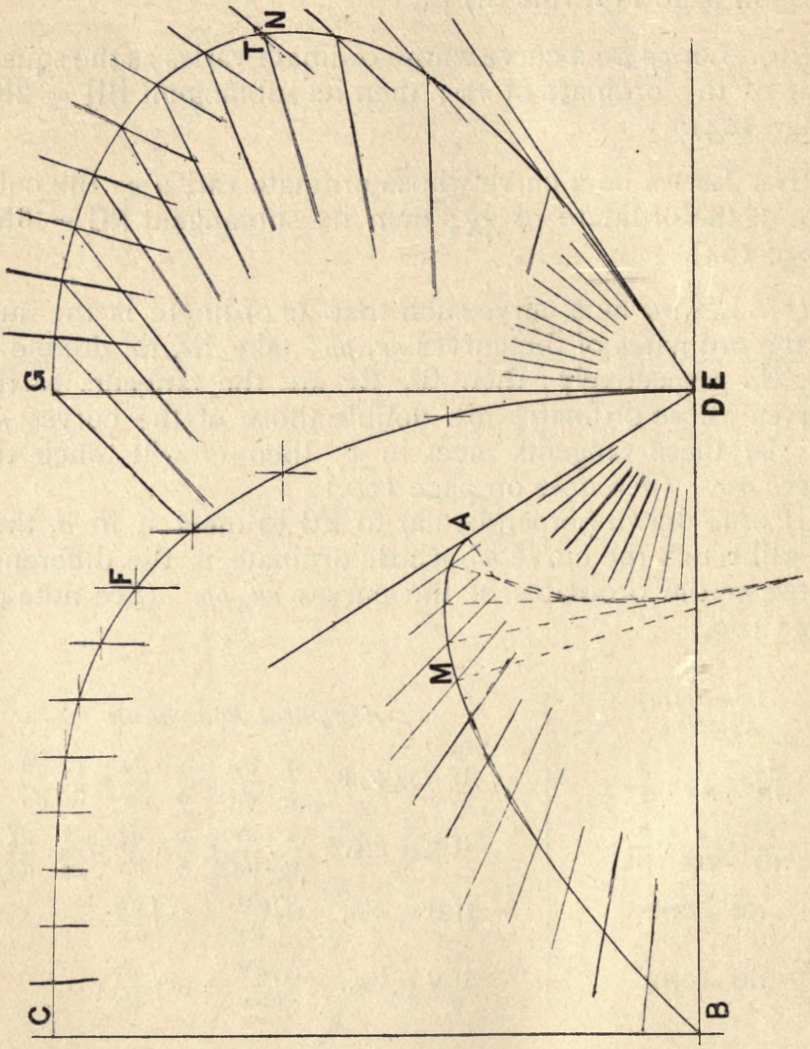
(iv). Let $\kappa\kappa$ be a curve whose ordinate varies as the cube root of the ordinate of $\chi\chi$; then its subtangent $NC = 3NQ$ (page 104).

(v). Let $\sigma\sigma$ be a curve such that its ordinate is the sum of the ordinates of the curves $\kappa\kappa$, $\rho\rho$; take Nc, Nr double of $N\kappa$, $N\rho$ respectively; then Cc, Rr are the tangents to the curves whose ordinates are double those of the curves $\kappa\kappa$, $\rho\rho$; let these tangents meet in s; then $s\sigma$ will touch the curve $\sigma\sigma$. (See note on page 100.)

If sd is drawn perpendicular to RC to meet it in d, then $d\delta$ will touch the curve $\delta\delta$, whose ordinate is the difference between the ordinates of the curves $\kappa\kappa$, $\rho\rho$. (See note on page 100.)

Geometrical Relations	*Analytical Equivalents*
$\frac{1}{NP} = \frac{1}{NF} + \frac{1}{NT}$	If $u = \phi\,.\,\theta$, $\frac{1}{u} \cdot \frac{du}{dx} = \frac{1}{\phi} \cdot \frac{d\phi}{dx} + \frac{1}{\theta} \cdot \frac{d\theta}{dx}$
$\frac{1}{NQ} = \frac{1}{NX} - \frac{1}{NL}$	If $v = \xi/\zeta$, $\frac{1}{v} \cdot \frac{dv}{dx} = \frac{1}{\xi} \cdot \frac{d\xi}{dx} - \frac{1}{\zeta} \cdot \frac{d\zeta}{dx}$
$NR = 2NP$	If $U = \sqrt{u}$, $U \Big/ \frac{dU}{dx} = 2u \Big/ \frac{du}{dx}$
$NC = 3NQ$	If $V = \sqrt[3]{v}$, $V \Big/ \frac{dV}{dx} = 3v \Big/ \frac{dv}{dx}$.

Hence
$$\frac{d}{dx}\Big[\sqrt{\{\phi(x)\,.\,\theta(x)\}} \pm \sqrt[3]{\{\xi(x)/\zeta(x)\}}\Big]$$
$$= \frac{U}{2u}\,\frac{du}{dx} \pm \frac{V}{3v} \cdot \frac{dv}{dx}$$
$$= \frac{\sqrt{(\phi\,.\,\theta)}}{2} \cdot \left\{\frac{1}{\phi} \cdot \frac{d\phi}{dx} + \frac{1}{\theta} \cdot \frac{d\theta}{dx}\right\} \pm \frac{\sqrt[3]{(\xi/\zeta)}}{3} \cdot \left\{\frac{1}{\xi} \cdot \frac{d\xi}{dx} - \frac{1}{\zeta} \cdot \frac{d\zeta}{dx}\right\}.$$


T
N
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DE
A
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M
C
B

II. The Area under any Curve

(Lect. XII, App. III, Prob. 5)

In the diagram on the opposite page, the curve CFD is a given curve, or a curve plotted to the rectangular axes BD, BC, which Barrow would be unable to integrate by any of the methods he has given, or, in fact, could give. The curve that I have chosen is one having the equation $y = \sqrt{(1 - x^4)}$, and the problem is to draw a curve that shall exhibit graphically the integral $\int dx/y$ for all values of the limits, subject to the condition that these limits must be positive numbers and not greater than unity. The first step is to construct the curve GNE, which is such that $r = \sqrt{(1 - \theta^4)}$, for which the method of construction is obvious from the diagram. Comparing this with the enunciation of Barrow's Prob. 5, the curve shown in the diagram is Barrow's curve turned through a right angle; thus, the point N is also the point T in Barrow's enunciation. Then a curve has to be constructed such that the several lines DN or DT are the respective subtangents. The curve produced is BMA, the method of construction being clearly shown in the figure; starting with B, each point is successively joined to its corresponding point on the curve GNE (so that MT is the tangent at M) and to the next point on GNE, and the point midway between the two points in which these cut the next ray from D is taken as the point on the curve BMA.

With this figure the area represented in Leibniz' notation by

$$\int_0^x 1/\sqrt{(1 - x^4)}\,dx \quad \text{or} \quad \int_0^\theta 1/\sqrt{(1 - \theta^4)}\,d\theta = \mathrm{R/DM} - \mathrm{R/DB};$$

for, if $r = f(\theta)$, since $\mathrm{DN} = r = f(\theta)$, from the curve GNE, and $\mathrm{DT} = \rho^2 \,.\, d\theta/d\rho$, from the curve BMA, it follows that

$$\int d\theta/f(\theta) = \int d\rho/\rho^2 = 1/\rho, \text{ for all limits.}$$

The value between the limits 0 and 1 works out as $1/\mathrm{DA} - 1/\mathrm{DB}$, which is found from the diagram to be $1/4{\cdot}8 - 1$, taking $\mathrm{DB} = 1$, that is $1{\cdot}304$; the true value is $\{\Gamma(1/2) \,.\, \Gamma(1/4)\} / \{4 \cdot \Gamma(3/4)\} = 1{\cdot}31$ about.

It is only suggested that this was the *purpose* of Barrow's problem, not that he *drew* such a figure as I have given.

III. Specimen Pages and Plate

Two specimen pages and a specimen of one of Barrow's plates here follow. The pages show the signs used by Barrow and the difficulty introduced by inconvenient or unusual notation, and by the method of "running on" the argument in one long string, with interpolations. The second page shows Barrow's algebraic symbolism. Especially note

$$k \,.\, m :: r \,.\, \frac{rm}{k} = \mathrm{EK}$$

which stands for

$$\text{since } k : m = r : \mathrm{EK}, \quad \therefore \mathrm{EK} = \frac{rm}{k}.$$

The specimen plate shows the quality of Barrow's diagrams. The most noticeable figure is Fig. 176, to be considered in connection with Newton's method as given in the *Principia*.

LECT. IX.

73

DEF concurrentes punctis S, T; erit ſemper $DT = 2\,DS$. Quòd ſi DE ſunt ut cubi ipſarum DF, erit ſemper $DT = 3\,DS$; ac ſimili deinceps modo. Fig. 99.

X. Sint rectæ VD, TB concurrentes in T, quas decuſſet poſitione data recta DB; tranſeant etiam per B lineæ EBE, FBF tales, ut ductâ quâcunque PG ad DB parallelâ, ſit perpetuò PF eodem ordine media Arithmeticè inter PG, PE; tangat autem BR curvam EBE; oportet lineæ FBF tangentem ad B determinare. Fig. 100.

Sumptis NM (ordinum in quibus ſunt PF, PE exponentibus) fiat $N \times TD \left.\begin{matrix} + M \\ - N \end{matrix}\right\} \times RD \,.\, M \times TD :: RD \,.\, SD$; & connectatur BS; hæc curvam FBF continget.

Nam utcunque ducta ſit PG, dictas lineas ſecans ut vides. Eſtque $EG \,.\, FG ::$ (*a*) $M \,.\, N$. ergo $FG \times TD \,.\, EG \times TD :: N \times TD \,.\, M \times TD$. Item $EF \times RD \,.\, EG \times TD :: M - N \times RD \,.\, M \times TD$. Quapropter (antecedentes conjungendo) erit $FG \times TD + EF \times RD \,.\, EG \times TD :: N \times TD + M - N \times RD \,.\, M \times TD$; (hoc eſt) :: (*b*) $RD \,.\, SD$. (*c*) Eſt autem $LG \times TD + KL \times RD \,.\, KG \times TD :: RD \,.\, SD$. quare $FG \times TD + EF \times RD \,.\, EG \times TD :: LG \times TD + KL \times RD \,.\, KG \times TD$. hinc, cùm ſit EG (*d*) $\sqsubset$ KG; erit $FG \times TD + EF \times RD \sqsubset LG \times TD + KL \times RD$; vel $FG \,.\, EF + TD \,.\, RD \sqsubset LG \,.\, KL + TD \,.\, RD$; ſeu (demptâ communi ratione) $FG \,.\, EF \sqsubset LG \,.\, KL$. vel componendo $EG \,.\, EF \sqsubset KG \,.\, KL$ (*e*) $\sqsubset EG \,.\, EL$. unde eſt $EF \sqsupset EL$. itaque punctum L extra curvam FBF ſitum eſt; adeoque liquet Propoſitum.

(*a*) 11. Lect. VII.

(*b*) Conſtr.

(*c*) 4. Lect. VII.

(*d*) Hyp.

(*e*) 1. Lect. VII.

XI. Quinetiam, reliquis ſtantibus iiſdem, ſi PF ſupponatur ejuſdem ordinis Geometricè media liquet (planè ſicut in modò præcedentibus) eandem BS curvam FBF contingere.

Exemplum. Si PF ſit è ſex mediis tertia, ſeu $M = 7$; & $N = 3$; erit $3\,TD + 4\,RD \,.\, 7\,MD :: RD \,.\, SD$; vel $SD = \dfrac{7MD \times RD}{3TD + 4RD}$.

XII. Patet etiam, accepto quolibet in curva FBF puncto (ceu F) rectam ad hoc tangentem conſimili pacto deſignari. Nempe per F ducatur recta PG ad DB parallela, ſecans curvam EBE ad E; & per E ducatur ER curvam EBE tangens; fiatque $N \times TP \left.\begin{matrix} + M \\ - N \end{matrix}\right\} \times RP$. Fig. 101.

ctanea, $bfa - 3mma = 3ffe - bme$; ſubſtituendóque $bfm - 3m^3 = 3fft - bmt$; ſeu, $\frac{bfm - 3m^3}{3ff - bm} = t$.

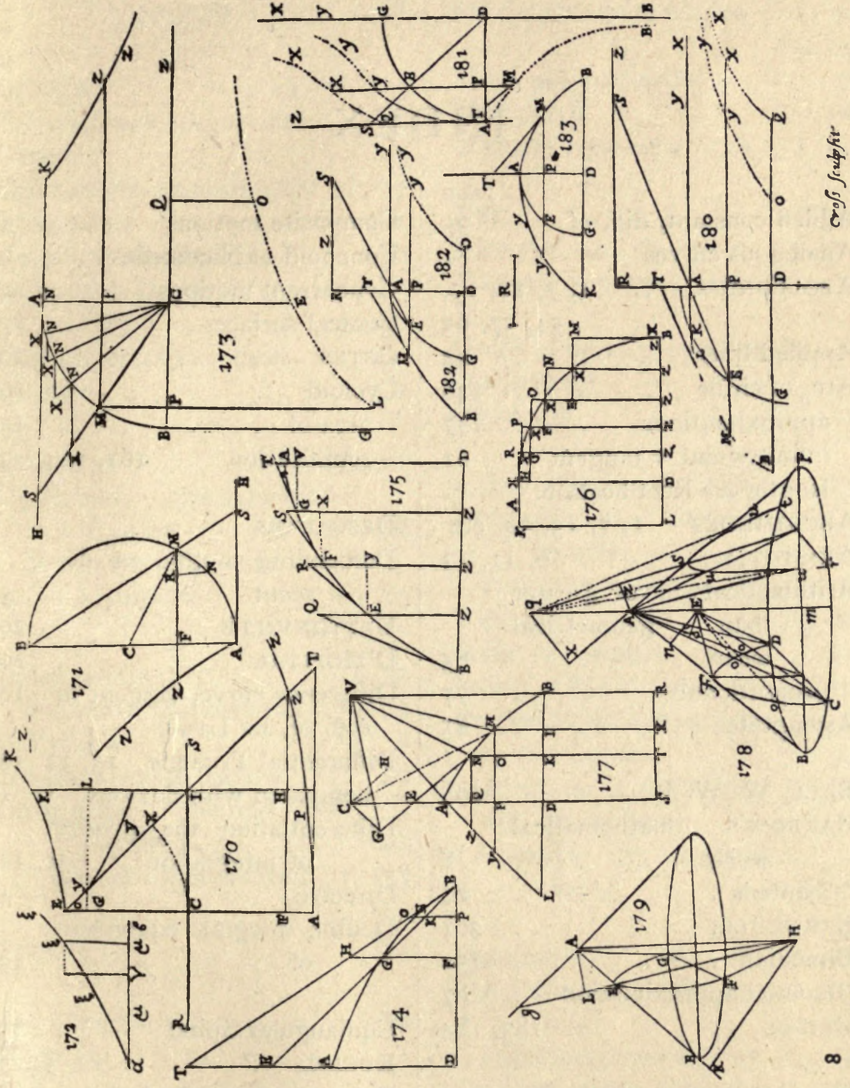
Exemp. IV.

Sit *Quadratrix* C M V (ad circulum C E B pertinens cui centrum A,) cujus axis V A; ordinatæ C A. M P ad V A perpendiculares.

Fig. 119.

Protractis rectis A M E, A N F, ductiſque rectis E K, F L ad A B perpendicularibus, dicantur arcus C B $= p$, radius A C $= r$; recta A P $= f$; A M $= k$. Eſtque jam C A arc. C B :: N R. arc. F E. hoc eſt, $r . p :: a . \frac{pa}{r} =$ arc. F E & A M. M P :: A E. E K; hoc eſt, $k . m :: r \frac{rm}{k} =$ E K; item A E. E K :: arc. F E. L K. hoc eſt $r . \frac{rm}{k} :: \frac{pa}{r} . \frac{pma}{rk} =$ L K. Verùm A M. A E :: A P. A K; hoc eſt $k . r :: f . \frac{rf}{k} =$ A K. ergò $\frac{rf}{k} - \frac{pma}{rk} =$ A L. Et $\frac{rrff}{kk} - \frac{2fmpa}{kk}$ (abjectis ſuperfluis) $=$ A L q; adeóque L F $q = \frac{rrkk - rrff + 2fmpa}{kk} = \frac{rrmm + 2fmpa}{kk}$.

Eſt autem A Q q. Q N q :: A L q. L F q; hoc eſt $Q : f - e . Q : m + a ::$ A L q. L F q. hoc eſt $ff - 2fe . mm + 2ma :: rrff - 2fmpa . rrmm + 2fmpa$. Unde (ſublatis ex norma rejectaneis) emerget *æquatio*, $ffpa + mmpa - rrfa = rrme$; ſeu $kkpa - rrfa = rrme$; vel ſubſtituendo juxta *præſcriptum*; $kkpm - rrfm = rrmt$; vel $\frac{kkp}{rr} - f = t$. Hinc colligitur eſſe rectam A T $= \frac{kk}{rr} p$; hoc eſt (quoniam, ut notum eſt, A V $= \frac{rr}{p}$) erit A T $= \frac{\text{AMq}}{\text{AV}}$; ſeu, A V. A M :: A M. A T.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PRINTED IN GREAT BRITAIN BY NEILL AND CO., LTD., EDINBURGH.